Group Theory I

1. Consider the set $H \equiv \{h_i, i = 1, 3\}$

$$h_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Construct the multiplication table for H and explain why it is not a group. What additions are necessary to make H into a group?

2. Consider the symmetric group $\mathcal{S}_3$. It has elements $\{p_{123}, p_{132}, p_{312}, p_{213}, p_{231}, p_{321}\}$, where

$$p_{ijk} \equiv \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$$

Construct the multiplication table and determine whether $\mathcal{S}_3$ is Abelian. What are the classes? What subgroups does $\mathcal{S}_3$ have?

3. Label the vertices of a square by the complex numbers $w_n = \mathrm{e}^{\mathrm{i}\pi(2n+1)/4}$, $n = 0, 1, 2, 3$. Rotations of the square can then be generated by multiplying the w_n by a complex number z. Find the z's which generate symmetries of the square and thus obtain the multiplication table of $\mathcal{C}_4$. What are the subgroups of $\mathcal{C}_4$? Can $\mathcal{C}_4$ be expressed as a direct product group?

Can all symmetries of the square be generated by multiplying the w_n by z? Explain why your answer is determined by the Abelian or non-Abelian nature of the groups involved.

4. Find the multiplication table of the direct product group $\mathcal{C}_2 \times \mathcal{C}_3$.

5. Show that the set $\{p_1, p_2, p_3\}$ (in the notation of the lecture notes) of elements of $\mathcal{D}_3$ may be written $\{e, r_1, r_2\}p_i$ or as $p_j\{e, r_1, r_2\}$ for suitable i, j. It follows that the set of elements of $\mathcal{D}_3$ may be written

$$\{e, r_1, r_2\}\{e, p_i\} \quad \text{or as} \quad \{e, p_j\}\{e, r_1, r_2\}.$$

Why is $\mathcal{D}_3$ nevertheless not a direct product group?

Are there any other right cosets of $\mathcal{D}_3$ other than $\{p_1, p_2, p_3\}$?

Group Theory II

1. ('90) Let $\mathcal{N} = \{n_1, \ldots, n_n\}$ be the set of elements of the group $\mathcal{G} = \{a_1, \ldots, a_g\}$ which commute with a particular element $\overline{a}$. Show that $\mathcal{N}$ is a subgroup of $\mathcal{G}$.

The set $\mathcal{N}a_i = \{n_1 a_i, \ldots, n_n a_i\}$ is the right coset of $\mathcal{N}$. Show that any two right cosets of $\mathcal{N}$ are either identical or have no elements in common.

Let $\{\mathcal{N}t_1, \ldots, \mathcal{N}t_h\}$ be a set of right cosets of $\mathcal{N}$ which together contain all the elements of $\mathcal{G}$ and are generated by the group elements $\{t_1, \ldots, t_h\}$. Show that the set of all elements conjugate to $\overline{a}$ contains h distinct elements which can be written as

$$\{t_1^{-1}\overline{a}t_1, \ldots, t_h^{-1}\overline{a}t_h\}.$$

Illustrate these result by considering the case $\mathcal{G} = \mathcal{D}_3$ (the dihedral group) and $\mathcal{N} = \{e, p_1\}$. Construct the three right cosets of $\mathcal{N}$ which together contain all the elements of $\mathcal{D}_3$, and obtain the elements conjugate to p_1. [You may use $p_2 = r_2 p_1 = p_1 r_1$ and $p_3 = r_1 p_1 = p_1 r_2$.]

2. Construct a four-dimensional representation of $\mathcal{C}_3$.

3. Reduce the 4-dimensional representation of the last question into its constituent irreps.

4. (i) Prove that the vector space spanned by x^n, yx^{n-1},..., y^n is $(n+1)$-dimensional. (ii) Reduce the representation of $\mathcal{C}_3$ provided by the x^n,... to its consituent irreps

[Hint: you may care to consider the transformation properties of the $n+1$ objects $z_p \equiv (x+{\rm i}y)^p(x-{\rm i}y)^{n+1-p}$.]

5. Using the basis (x^3, x^2y, y^2x, y^3) construct a 4-dimensional representation of $\mathcal{D}_3$. Reduce it to irreps.

Group Theory III

1. Calculate the characters for the 4-dimensional representation of $\mathcal{D}_3$ that you found in the previous problem set. Hence reduce that representation into irreps of $\mathcal{D}_3$.

Find the reduction of the 6-dimensional representation in which the classes $(E, 2C_3, 3C_2)$ have characters $(6, 3, -2)$.

2. ('91) The tetrahedral group $\mathcal{T}$ is of order 12 and can be generated by taking powers and products of

$$R_1 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \;, \quad R_2 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \;.$$

Calculate the orders of R_1 and R_2. Interpret the results geometrically. Find two elements of $\mathcal{T}$ which are conjugate to R_1.

An electron in an atom has 9 degenerate energy eigen-states with wave-functions

$$\begin{aligned} \phi_i &= x_i f(r) \\ \psi_{ij} &= x_i x_j g(r) \end{aligned} \qquad i, j = 1, 2, 3,$$

where f and g are functions of the radial coordinate r. These form a basis for a 9-dimensional rep of $\mathcal{T}$. Find linear combinations which form a basis for irreps of $\mathcal{T}$.

The system is perturbed by the additional potential $V = V_0 xyz$. To what degree is the degeneracy removed?

3. ('90) A particle moves in the (x, y) plane in a potential with dihedral symmetry $\mathcal{D}_4$ (and symmetry axes aligned with the coordinate axes). The wave-functions of degenerate energy eigen-states are $\psi_x = xf(r)$ and $\psi_y = yf(r)$, where $r = \sqrt{x^2 + y^2}$. Find the eight 2×2 matrices which describe how these states transform under $\mathcal{D}_4$. Show how the elements of $\mathcal{D}_4$ are divided into conjugacy classes.

Calculate the characters of the 2×2 rep of $\mathcal{D}_4$ obtained in the first part of this question. Show that they are consistent with the irreducibility of the rep. Use the orthonormality relations of characters to obtain a character table for all the irreps of $\mathcal{D}_4$.

Two particles of equal mass placed in the states ψ_x and ψ_y interact through a potential which preserves $\mathcal{D}_4$ symmetry. To which irreps of $\mathcal{D}_4$ do the resulting states belong?

4. Show that

$$\frac{\sin(j_1 + \frac{1}{2})\phi \, \sin(j_2 + \frac{1}{2})\phi}{\sin^2 \frac{1}{2}\phi} = \frac{\sin(j_1 + j_2 + \frac{1}{2})\phi}{\sin \frac{1}{2}\phi} + \frac{\sin(j_1\phi)\sin(j_2\phi)}{\sin^2 \frac{1}{2}\phi}.$$

Hence, or otherwise, show that the character $\chi^{(j)}(\phi)$ of a rotation through angle ϕ in the spin-j irrep of $SU(2)$ satisfies

$$\chi^{(j_1)}(\phi)\chi^{(j_2)}(\phi) = \chi^{(j_1+j_2)}(\phi) + \chi^{(j_1+j_2-1)}(\phi) + \cdots + \chi^{(|j_1-j_2|)}(\phi).$$

Explain the significance of this result for the decomposition into irreps of the representation $D^{(j_1)} \times D^{(j_2)}$ of $SU(2)$.

5. A system has symmetry group O. Perturbations are applied which reduce the symmetry to (i) $\mathcal{T}$, (ii) $\mathcal{C}_{3v}$, (iii) $\mathcal{C}_4$. In each case find how energy levels belonging to the irreps E, T_1 and T_2 of O are split by the perturbation.

6. Find the selection rules for electric dipole transitions when the symmetry group of the unperturbed Hamiltonian is (i) $\mathcal{D}_3$, (ii) O.

7. Show that when the complex 2-vector $\boldsymbol{\eta}$ is defined as in Box 2, we have

$$x = \boldsymbol{\eta}^\dagger \cdot \boldsymbol{\sigma}_x \cdot \boldsymbol{\eta} \qquad y = \boldsymbol{\eta}^\dagger \cdot \boldsymbol{\sigma}_y \cdot \boldsymbol{\eta} \qquad z = \boldsymbol{\eta}^\dagger \cdot \boldsymbol{\sigma}_z \cdot \boldsymbol{\eta}$$

where $\boldsymbol{\eta}^\dagger$ is the complex-conjugate-transpose of $\boldsymbol{\eta}$ and

$$\boldsymbol{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \boldsymbol{\sigma}_y \equiv \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad ; \quad \boldsymbol{\sigma}_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli spin matrices.

8. Show that "rotating" $\boldsymbol{\eta}$ with the matrix

$$s_z(\phi) \equiv \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \\ 0 & e^{\mathrm{i}\phi/2} \end{pmatrix}$$

has the effect of rotating the (x, y, z) coordinates through ϕ about the z axis. What happens to $\boldsymbol{\eta}$ when the (x, y, z) axes are rotated through 2π?

Show that when $\boldsymbol{\eta}$ suffers an infinitesimal "rotation"

$$\delta\boldsymbol{\eta} = \tfrac{\mathrm{i}}{2}\delta\theta\, \boldsymbol{\sigma}_x \cdot \boldsymbol{\eta},$$

the point $\mathbf{r} = (x, y, z)$ is rotated by $\delta\theta$ about the x axis.

9. Verify that $|\mathbf{r}_1 - \mathbf{r}_2|$ is invariant when the $\mathbf{r}_i$ are mapped into new vectors $\mathbf{r}'_i$ by the transformation generated by a unitary matrix $\mathbf{M}$ acting on Weyl spinors $\boldsymbol{\eta}_i$ as described in Box 2.

10. What transformation of the vector $\boldsymbol{\eta}$ generates the transformation $\mathbf{r} \to -\mathbf{r}$?

11. Explain the connection between Box 2 and Example 14.