Further Quantum Mechanics HT 2014
Problems 2 (Easter Vacation)

Radiative transitions

2.1* Let $|E, l, m\rangle$ denote a stationary state of an atom with orbital angular-momentum quantum numbers l, m, and let $x_\pm = x \pm \mathrm{i}y$ be complex position operators while $L_\pm = L_x \pm \mathrm{i}L_y$ are the usual orbital angular-momentum ladder operators. Show that $x_\pm|E, l, m\rangle$ is an eigenket of L_z with eigenvalue $m \pm 1$. Show also that

$$[L_+, x_+] = [L_-, x_-] = 0 \quad \text{and} \quad [L_+, x_-] = -[L_-, x_+] = 2z.$$

Hence show that

$$\langle E', l', m|z|E, l, m\rangle = \alpha_+(l, m)\langle E', l', m|x|E, l, m+1\rangle - \alpha_-(l', m)\langle E', l', m-1|x|E, l, m\rangle.$$

where $\alpha_\pm(l, m) \equiv \sqrt{l(l+1) - m(m \pm 1)}$. [Hint: compute $\langle E', l', m|x|E, l, m+1\rangle$]

2.2 Given that $a_0 = \hbar/(\alpha m_\mathrm{e} c)$ show that the product $a_0 k$ of the Bohr radius and the wavenumber of a photon of energy E satisfies

$$a_0 k = \frac{E}{\alpha m_\mathrm{e} c^2}. \tag{2.1}$$

Hence show that the wavenumber k_α of an Hα photon satisfies $a_0 k_\alpha = \frac{5}{72}\alpha$ and determine λ_α/a_0. What is the connection between this result and our estimate that $\sim 10^7$ oscillations are required to complete a radiative decay. Does it imply anything about the way the widths of spectral lines from allowed atomic transitions vary with frequency?

2.3 Given that a system's Hamiltonian is of the form

$$H = \frac{p^2}{2m_\mathrm{e}} + V(\mathbf{x}) \tag{2.2}$$

show that $[x, [H, x]] = \hbar^2/m_\mathrm{e}$. By taking the expectation value of this expression in the state $|k\rangle$, show that

$$\sum_{n \neq k} |\langle n|x|k\rangle|^2 (E_n - E_k) = \frac{\hbar^2}{2m_\mathrm{e}}, \tag{2.3}$$

where the sum runs over all the other stationary states.

The **oscillator strength** of a radiative transition $|k\rangle \to |n\rangle$ is defined to be

$$f_{kn} \equiv \frac{2m_\mathrm{e}}{\hbar^2}(E_n - E_k)|\langle n|x|k\rangle|^2. \tag{2.4}$$

Show that $\sum_n f_{kn} = 1$. What is the significance of oscillator strengths for the allowed radiative transition rates of atoms?

2.4 With $|nlm\rangle$ a stationary state of hydrogen, which of these matrix elements is non-zero?

$$\begin{array}{ccc} \langle 100|z|200\rangle & \langle 100|z|210\rangle & \langle 100|z|211\rangle \\ \langle 100|z|300\rangle & \langle 100|z|310\rangle & \langle 100|z|320\rangle \\ \langle 100|x|200\rangle & \langle 100|x|210\rangle & \langle 100|x|211\rangle \end{array}$$

2.5 With $|nlm\rangle$ a stationary state of hydrogen and given that

$$\langle \mathbf{x}|100\rangle = \frac{2\mathrm{e}^{-r/a_0}}{a_0^{3/2}}\mathrm{Y}_0^0; \quad \langle \mathbf{x}|210\rangle = \frac{r}{a_0}\frac{\mathrm{e}^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}}\mathrm{Y}_1^0(\theta, \phi); \quad \mathrm{Y}_1^0(\theta, \phi) = \sqrt{\frac{6}{8\pi}}\cos\theta,$$

show that

$$\langle 100|z|210\rangle = 2^{5/2}(2/3)^5 a_0$$

Hence show that Einstein's A coefficient for the Lyman α transition is

$$A = \tfrac{1}{3}2^4(2/3)^8\pi\alpha^3\nu.$$

[Hint: recall that $\mathcal{R} = \frac{1}{2}\alpha^2 m_\mathrm{e} c^2$ and $a_0 = \hbar/\alpha m_\mathrm{e} c$.]

What is the characteristic timescale in ns for the radiative decay of an isolated hydrogen atom that starts from the $n = 2$ level?

2.6 With $|nlm\rangle$ a stationary state of hydrogen, and given that

$$\mathrm{Y}_1^0(\theta,\phi)=\sqrt{\frac{6}{8\pi}}\cos\theta \quad \mathrm{Y}_1^1(\theta,\phi)=-\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} \quad \mathrm{Y}_1^{-1}(\theta,\phi)=\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi},$$

show that

$$\begin{aligned}\langle 100|(x-\mathrm{i}y)|211\rangle &= -\sqrt{2}\langle 100|z|210\rangle\\ \langle 100|(x-\mathrm{i}y)|21-1\rangle &= 0.\end{aligned}$$

Write down the values of $\langle 100|(x+\mathrm{i}y)|21-1\rangle$ and $\langle 100|(x+\mathrm{i}y)|211\rangle$ and hence show that with

$$|\psi\rangle\equiv\frac{1}{\sqrt{2}}(|211\rangle-|21-1\rangle),$$

$\langle 100|x|\psi\rangle=-\langle 100|z|210\rangle$. Explain the physical significance of this result.

2.7 Derive the selection rules

$$\begin{aligned}\langle n'l'm'|x_+|nlm\rangle &= 0 \quad \text{unless} \quad m'=m+1\\ \langle n'l'm'|x_-|nlm\rangle &= 0 \quad \text{unless} \quad m'=m-1.\end{aligned}$$

where $x_\pm=x\pm\mathrm{i}y$. From this selection rule one infers that when the atom sits in a magnetic field along the z axis and the spectrometer looks along the z axis, the detected photons will be circularly polarised. Show that linearly polarised photons *can* be detected from an atom that's in a magnetic field.

From the above rules it might be argued that photons emitted along the z axis will be circularly polarised even in the absence of a magnetic field. Why is this argument bogus?