GEOMETRY & PHYSICS: PROBLEMS

1. Let M_1, M_2 and M_3 be manifolds (not necessarily of the same dimension) and let $\alpha : M_1 \to M_2$ and $\beta : M_2 \to M_3$, be C^∞ maps between them. Show that the induced Jacobian maps α_* etc satisfy $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$.

2. Show that $\mathrm{d}(fg) = f(m)\mathrm{d}g + g(m)\mathrm{d}f$ for $f, g \in \mathcal{F}_m$.

3. Let X, Y and Z be vector fields defined in a neighbourhood of m, and such that $Z_m = 0$, and let ω be a 1-form near m. Prove that

$$\mathrm{d}\omega(X + Z, Y) = \mathrm{d}\omega(X, Y)$$

and explain the significance of this result.

4. Consider a manifold with a symplectic form ω. Let $\mathcal{F}$ and $\mathcal{G}$ be the vector fields that ω associates with the functions f and g, respectively. Show that the Lie bracket $[\mathcal{F}, \mathcal{G}]$ is the vector field associated with the Poisson bracket $\{f, g\}$.

5. Show that the metric tensor of a Riemannian manifold satisfies the Schwartz inequality, $g(X,Y)^2 \leq g(X,X)g(Y,Y)$, and the triangle inequality, $g(X+Y, X+Y)^{1/2} \leq g(X,X)^{1/2} + g(Y,Y)^{1/2}$.

6. Let Ω be the natural volume n-form of an orientable, n-dimensional, Riemannian manifold and $\partial/\partial\phi^i$ be an arbitrary set of base vectors. Show that

$$\Omega = \left|\det(g_{ij})\right|^{1/2} \mathrm{d}\phi^1 \wedge \cdots \wedge \mathrm{d}\phi^n.$$

Hence show that $\Omega_{ij\ldots k} = \left|\det(g_{ab})\right|^{1/2} \epsilon_{ij\ldots k}$, where ϵ is the usual Levi-Civita symbol.

7. Show that in 3d Euclidean space, with A and B 1-forms

$$\begin{aligned} A \times B &= *(A \wedge B) \\ \nabla \times A &= *\mathrm{d}A \\ \nabla \cdot A &= \mathrm{d} * A. \end{aligned}$$