Complex Numbers & ODEs: Set 3

Sections A: Easy Pieces

1. Find the solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3e^x$$

for which $y = 3$ and $\frac{dy}{dx} = 0$ at $x = 0$.

2. Deduce that the o.d.e.

$$\frac{dy}{dx} + \frac{2x}{y} = 3$$

is satisfied when $A(y - x) = (2x - y)^2$, where A is an arbitrary constant.

3. Deduce that the o.d.e.

$$2\frac{dy}{dx} = \frac{y}{x} + \frac{y^3}{x^3}$$

is solved when $x = A(1 - x^2/y^2)$, where A is an arbitrary constant.

4. Deduce that the o.d.e.

$$xy\frac{dy}{dx} - y^2 = (x + y)^2 e^{-y/x}$$

is solved when $\ln x = e^{y/x}/(1 + y/x) + A$, where A is an arbitrary constant.

5. Show that the general solution of the o.d.e.

$$\frac{dy}{dx} = \frac{x - y}{x - y + 1}$$

is $y = x + 1 - \sqrt{2(x + A)}$, where A is an arbitrary constant.

6. By introducing a new variable $Y = (4y - x)$, or otherwise, show that the solution of the o.d.e.

$$\frac{dy}{dx} - 16y^2 + 8xy = x^2$$

satisfies $4y - x - \frac{1}{2} = A(4y - x + \frac{1}{2})e^{4x}$, where A is an arbitrary constant.

7. Solve the o.d.e.

$$\frac{dy}{dx} = \frac{(3x^2 + 2xy + y^2)\sin x - (6x + 2y)\cos x}{(2x + 2y)\cos x}.$$

[Hint: look for a function $f(x, y)$ whose differential df gives the o.d.e.]

8. By using the substitution $y = z^2$, or otherwise, find the general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x^3 y^{1/2}.$$

9. The currents i_1 and i_2 in two coupled LC circuits satisfy the equations

$$\begin{aligned} L\frac{\mathrm{d}^2 i_1}{\mathrm{d}t^2} + \frac{i_1}{C} - M\frac{\mathrm{d}^2 i_2}{\mathrm{d}t^2} &= 0 \\ L\frac{\mathrm{d}^2 i_2}{\mathrm{d}t^2} + \frac{i_2}{C} - M\frac{\mathrm{d}^2 i_1}{\mathrm{d}t^2} &= 0 \ , \end{aligned}$$

where $0 < M < L$. Find formulae for the two possible frequencies at which the coupled system may oscillate sinusoidally. [Hint: obtained uncoupled equations by taking the sum and difference of the given equations.]

Section B: more challenging problems

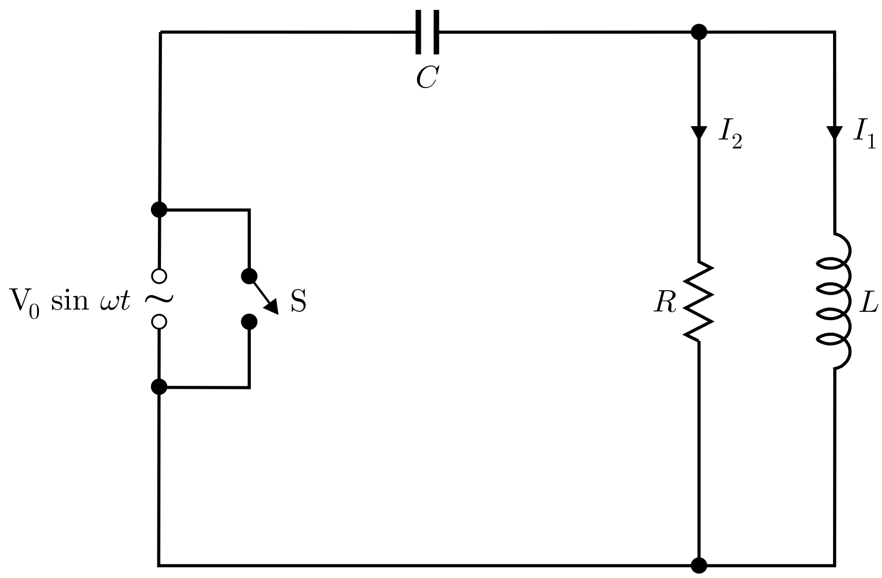
10. The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + ky = y^n \sin x,$$

where k and n are constants, is linear and homogeneous for $n = 1$. State a property of the solutions to this equation for $n = 1$ that is **not** true for $n \neq 1$.

Solve the equation for $n \neq 1$ by making the substitution $z = y^{1-n}$.

11. An alternating voltage $V = V_0 \sin \omega t$ is applied to the circuit below.



The following equations may be derived from Kirchoff's laws:

$$I_2 R + \frac{Q}{C} = V,$$
$$L\frac{\mathrm{d}I_1}{\mathrm{d}t} = I_2 R,$$
$$\frac{\mathrm{d}Q}{\mathrm{d}t} = I_1 + I_2,$$

where Q is the charge on the capacitor.

Derive a second-order differential equation for I_1, and hence obtain the steady state solution for I_1 after transients have decayed away.

Determine the angular frequency ω at which I_1 is in phase with V, and obtain expressions for the amplitudes of I_1 and I_2 at this frequency.

Suppose now that the switch S is closed and the voltage supply removed when I_1 is at its maximum value. Obtain the solution for the subsequent variation of I_1 with time for the case $L = 4CR^2$, and sketch the form of your solution.