Complex Numbers & ODEs: Set 2

Sections A: Easy Pieces

1. Solve the o.d.e.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y\cos x}{\sin x}.$$

2. Solve the o.d.e.

$$x(x-1)\frac{\mathrm{d}y}{\mathrm{d}x} + y = x(x-1)^2.$$

3. Solve the o.d.e.

$$2x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2.$$

4. L_1 is the differential operator

$$L_1 = \left(\frac{\mathrm{d}}{\mathrm{d}x} + 2\right).$$

Evaluate (i) $L_1 x^2$, (ii) $L_1\big(x\mathrm{e}^{2x}\big)$, (iii) $L_1\big(x\mathrm{e}^{-2x}\big)$.

5. L_2 is the differential operator

$$L_2 = \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right).$$

Express the operator $L_3 = L_2 L_1$ in terms of $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$, $\dfrac{\mathrm{d}}{\mathrm{d}x}$, etc. Show that $L_1 L_2 = L_2 L_1$.

6. Find the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 10\cos x.$$

7. Show that the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2\mathrm{e}^{-x} + x^3 + 2\cos x.$$

is $y = (A + Bx + x^2)\mathrm{e}^{-x} + \sin x + x^3 - 6x^2 + 18x - 24$, where A, B are arbitrary constants.

8. Solve the simultaneous differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}z}{\mathrm{d}x} + 4y + 10z - 2 = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}z}{\mathrm{d}x} + y - z + 3 = 0,$$

where $y = 0$ and $z = -2$ when $x = 0$.

Section B: more challenging problems

9. A mass m is constrained to move in a straight line and is attached to a spring of strength $\lambda^2 m$ and a dashpot which produces a retarding force $-\alpha m v$, where v is the velocity of the mass. Find the displacement of the mass when an amplitude-modulated periodic force $Am\cos pt\sin\omega t$ with $p \ll \omega$ and $\alpha \ll \omega$ is applied to it.

Show that for $\omega = \lambda$ the displacement is the amplitude-modulated wave

$$= -2\frac{\cos\omega t\sin(pt+\phi)}{\sqrt{4\omega^2p^2+\alpha^2\omega^2}} \quad \text{where} \quad \cos\phi = \frac{2\omega p}{\sqrt{4\omega^2p^2+\alpha^2\omega^2}}.$$

10. Solve the differential equations

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}z}{\mathrm{d}x} + 3y + z = \mathrm{e}^{2x}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}z}{\mathrm{d}x} + 2y - z = 0.$$

Is it possible to have a solution to these equations for which $y = z = 0$ when $x = 0$?

11. When a varying couple $I\cos nt$ is applied to a torsional pendulum with natural period $2\pi/m$ and the moment of inertia I, the angle of the pendulum satisfies the equation of motion $\ddot\theta + m^2\theta = \cos nt$. The couple is first applied at $t = 0$ when the pendulum is at rest in equilibrium. Show that in the subsequent motion the root mean square angular displacement is $1/|m^2 - n^2|$ when the average is taken over a time large compared with $1/|m-n|$. Discuss the motion as $|m-n| \to 0$.

12. Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + (\beta^2+1)y = \mathrm{e}^x\sin^2 x$$

for general values of the real parameter β. Explain why this solution fails for $\beta = 0$ and $\beta = 2$ and find solutions for these values of β.

13. Verify that $y = x + 1$ is a solution of

$$(x^2-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0.$$

Writing $y = (x+1)u$, show that $u' = \mathrm{d}u/\mathrm{d}x$ satisfies

$$\frac{\mathrm{d}u'}{\mathrm{d}x} + \frac{3x-1}{x^2-1}u' = 0.$$

Hence show that the general solution of the original equation is

$$y = K\left(\tfrac{1}{4}(x+1)\ln\frac{x-1}{x+1} - \tfrac{1}{2}\right) + K'(x+1),$$

where K and K' are arbitrary constants.