Complex Numbers & ODEs: Set 1

Sections A: Easy Pieces

1. For $z = x + \mathrm{i}y$ find the real and imaginary parts of

$$\text{(i) } 2+z; \quad \text{(ii) } z^2; \quad \text{(iii) } z^*; \quad \text{(iv) } 1/z; \quad \text{(v) } |z|$$

2. Use the representation $z = |z|\mathrm{e}^{\mathrm{i}\arg(z)}$ to evaluate

$$\text{(i) } |1+\mathrm{i}|; \quad \text{(ii) } \arg(1+\mathrm{i}); \quad \text{(iii) } \arg\left(\frac{1+\mathrm{i}}{1-\mathrm{i}}\right); \quad \text{(iv) } \left|\frac{2+3\mathrm{i}}{5+\mathrm{i}}\right|.$$

3. For $z = x + \mathrm{i}y$ sketch the curves (i) $|z| = 1$, (ii) $\Re\mathrm{e}(z) = \frac{1}{2}$, (iii) $z = t\mathrm{e}^{\mathrm{i}t}$ (for real values of the parameter t) in the Argand diagram for z.

4. Use de Moivre's theorem and the resulting identity $\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2}$ to write the following in the form $a + \mathrm{i}b$, where a and b are real:
(i) e^{i}; (ii) $\sqrt{\mathrm{i}}$; (iii) $\ln \mathrm{i}$; (iv) $\cos \mathrm{i}$; (v) $\sin \mathrm{i}$; (vi) $\sinh(x + \mathrm{i}y)$.

5. The complex numbers a, b and c represent points in the Argand diagram. Give a geometrical interpretation of $|a - b|$ and $\arg[(a - b)/(a - c)]$.

6. Find all the solutions of the equation $z^n = 1$, where n is a positive integer.

7. Prove that the sum and product of the roots x_i of the polynomial $a_n x^n + \cdots + a_0$ satisfy $\sum z_i = -a_{n-1}/a_n$ and $\prod x_i = (-1)^n a_0/a_n$. Hence find the sum and the product of the roots of $P = x^3 - 6x^2 + 11x - 6$. Show that $x = 1$ is a root and by writing $P = (x - 1)Q$, where Q is a quadratic, find the other two roots. Verify that the roots have the expected sum and product.

Section B: more challenging problems

8. Sketch the curves C_1 and C_2 in the Argand diagram for z defined respectively by $\arg[(z - 4)/(z - 1)] = \pi/2$ and $\arg[(z - 4)/(z - 1)] = 3\pi/2$.

9. By noting that $\mathrm{e}^{\mathrm{i}5\theta} = \big(\cos\theta + \mathrm{i}\sin\theta\big)^5$, express $\cos 5\theta$ as a polynomial in $\cos\theta$.

10. Show that

$$\sum_{n=0}^{\infty} 2^{-n} \cos n\theta = \frac{1 - \frac{1}{2}\cos\theta}{\frac{5}{4} - \cos\theta}.$$

11. Show that the equation $(z + \mathrm{i})^n - (z - \mathrm{i})^n = 0$ has roots $z = \cot(r\pi/n)$, where $r = 1, 2, \ldots, n - 1$ and show that $\cot^2 \frac{1}{5}\pi + \cot^2 \frac{2}{5}\pi = 2$.

12. Find the roots of the equation $(z-1)^n + (z+1)^n = 0$. Hence solve the equation $x^3 + 15x^2 + 15x + 1 = 0$.

13. Prove that

$$\sum_{r=1}^{n} \binom{n}{r} \sin 2r\theta = 2^n \sin n\theta \cos^n \theta \quad \text{where} \quad \binom{n}{r} \equiv \frac{n!}{(n-r)!r!}.$$

[Hint: express the left side as $\Im\mathrm{m}\left(\mathrm{e}^{in\theta} \sum \binom{n}{r} \mathrm{e}^{\mathrm{i}(2r-n)\theta} \right)$.]

14. Show that the equation $(z+1)^n - \mathrm{e}^{2in\theta}(z-1)^n = 0$ has root $z = -\mathrm{i}\cot(\theta + r\pi/n)$. Show that

$$\prod_{r=1}^{n} \cot\left(\theta + \frac{r\pi}{n}\right) = \begin{cases} (-1)^{n/2} & \text{for } n \text{ even} \\ (-1)^{(n+1)/2} \cot n\theta & \text{for } n \text{ odd.} \end{cases}$$

15. Find all the roots, real and complex, of the equation $z^3 - 1 = 0$. If ω is one of the complex roots, prove that $1 + \omega + \omega^2 = 0$. Find the sums of the following series:

$$S_1 = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots; \quad S_2 = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots; \quad S_3 = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

[Hint: note that $S_1 + S_2 + S_3 = \mathrm{e}^x$ and calculate $\mathrm{e}^{\omega x}$ and $\mathrm{e}^{\omega^2 x}$.]

16. Show that $\cos 2n\theta$ can be expressed as a ploynomial in $s \equiv \sin^2\theta$, namely $\cos 2n\theta = 1 + a_1 s + a_2 s^2 + \cdots + a_n s^n$, where n is a positive integer.

Hence show that

$$\cos 2n\theta = \prod_{r=1}^{n} \left\{ 1 - \frac{\sin^2\theta}{\sin^2[\frac{1}{4}(2r-1)\pi/n]} \right\}.$$