Solutions Problem Set 1

1. In polar coords, radial speed is $\dot{r}$, tangential speed is $r\dot{\phi}$, so $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$ and

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V.$$

Hence $\partial L/\partial\dot{r} = m\dot{r}$ and $\partial L/\partial r = mr\dot{\phi}^2 - \partial V/\partial r$, so EL equation for r is

$$m\frac{\mathrm{d}\dot{r}}{\mathrm{d}t} - mr\dot{\phi}^2 + \frac{\partial V}{\partial r} = 0$$

Similarly, $\partial L/\partial\dot{\phi} = mr^2\dot{\phi}$, so the EL equation for ϕ is

$$m\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\phi}) + \frac{\partial V}{\partial \phi} = 0.$$

If the potential is axisymmetric, $\partial V/\partial\phi = 0$, so last equation states that the angular momentum $mr^2\dot{\phi}$ is constant. If the motion is circular, $\dot{r} = 0 = \ddot{r}$ and the radial equation becomes

$$-m\frac{v^2}{r} = -\frac{\partial V}{\partial r},$$

where $v = r\dot{\phi}$ is the speed. Thus the force $-\partial V/\partial r$ is equal to m times the centripetal acceleration $-v^2/r$.

2. The equation of motion is

$$m\ddot{\mathbf{r}} = -\nabla V(r) = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}.$$

Crossing through with $\mathbf{r}$ we get $m\mathbf{r}\times\ddot{\mathbf{r}} = 0$. But

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times\dot{\mathbf{r}}) = (\dot{\mathbf{r}}\times\dot{\mathbf{r}}) + (\mathbf{r}\times\ddot{\mathbf{r}}) = (\mathbf{r}\times\ddot{\mathbf{r}})$$

so the equation of motion states that the vector $\mathbf{r}\times\dot{\mathbf{r}}$ is constant. This result implies that the motion is confined to the plane containing the initial values of $\mathbf{r}$ and $\dot{\mathbf{r}}$.

With (r,ϕ) polar coords in this plane, the Lagrangian is $L = \frac{1}{2}m[\dot{r}^2 + (r\dot{\phi})^2] - V$. Now

$$r \equiv \frac{1}{u} \quad\Rightarrow\quad \dot{r} = -\frac{\dot{u}}{u^2}$$

and L becomes

$$L = \tfrac{1}{2}m\left(\frac{\dot{u}^2}{u^4} + \frac{\dot{\phi}^2}{u^2}\right) - V(u).$$

Given $V = -\alpha u$ the EL eqn for u becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\dot{u}}{u^4}\right) + 2m\frac{\dot{u}^2}{u^5} + m\frac{\dot{\phi}^2}{u^3} - \alpha = 0,$$

while the EL eqn for ϕ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\phi}}{u^2}\right) = 0 \quad\Rightarrow\quad \frac{\dot{\phi}}{u^2} = h,\ \text{a constant}$$

This results yields $\mathrm{d}\phi = hu^2\,\mathrm{d}t$, which we can use to eliminate t from the u equation:

$$hu^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{m}{u^4}hu^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\right) + \frac{2mh^2u^4}{u^5}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + mh^2u - \alpha = 0$$

When we expand the derivative on the left we get a term

$$-\frac{2mh^2}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2$$

that cancels the second term, and the equation cleans up to

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u=\frac{\alpha}{mh^2}.$$

The GS of this is

$$u=A\cos(\phi-\phi_0)+\frac{\alpha}{mh^2}.$$

The orbit is bound if u cannot reach zero, i.e., if $\alpha/mh^2 > A$. Defining $x = r\cos(\phi-\phi_0)$ and $y = r\sin(\phi-\phi_0)$, we have

$$1=rA\cos(\phi-\phi_0)+\frac{r\alpha}{mh^2}\quad\Rightarrow\quad 1=xA+\frac{r\alpha}{mh^2}$$

so

$$1-2Ax+A^2x^2=\frac{\alpha^2}{m^2h^4}(x^2+y^2)\quad\Leftrightarrow\quad x^2\left(\frac{\alpha^2}{m^2h^4}-A^2\right)+2Ax+y^2=1,$$

which is the equation of an ellipse or hyperbola depending on the sign of the coeff of x^2.

3. In spherical polars the kinetic energy is

$$T=\tfrac{1}{2}m\left[\dot r^2+(r\dot\theta)^2+(r\sin\theta\dot\phi)^2\right]\quad\Rightarrow\quad L=\tfrac{1}{2}m\left[\dot r^2+(r\dot\theta)^2+(r\sin\theta\dot\phi)^2\right]-m\Phi$$

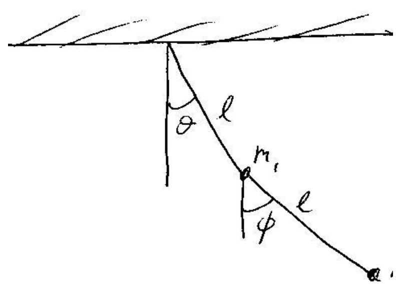
so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\dot r-r\dot\theta^2-r\sin^2\theta\dot\phi^2+\frac{\partial\Phi}{\partial r}&=0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot\theta\right)-r^2\sin\theta\cos\theta\dot\phi^2+\frac{\partial\Phi}{\partial\theta}&=0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\sin^2\theta\dot\phi\right)+\frac{\partial\Phi}{\partial\phi}&=0\end{aligned}$$

Now we calculate the derivative of the squared angular momentum, which is the sum of contributions $r^2\dot\theta$ perpendicular to the meridional plane and $r^2\sin\theta\dot\phi$ parallel to the z axis:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[(r^2\dot\theta)(r^2\dot\theta)+\frac{(r^2\sin^2\theta\dot\phi)^2}{\sin^2\theta}\right]&=2r^2\dot\theta r^2\sin\theta\cos\theta\dot\phi^2-2\frac{(r^2\sin^2\theta\dot\phi)^2}{\sin^3\theta}\cos\theta\dot\theta\\ &=2r^4\dot\theta\sin\theta\cos\theta\dot\phi^2-2r^4\sin\theta\dot\phi^2\cos\theta\dot\theta=0\end{aligned}$$

4.



From the figure

$$\begin{aligned}T&=\tfrac{1}{2}m_1(l\dot\theta)^2+\tfrac{1}{2}m_2(l\dot\theta+l\dot\phi)^2\\ V&=-m_1gl\cos\theta-m_2g(l\cos\theta+l\cos\phi)\\ &\simeq gl(m_1\tfrac{1}{2}\theta^2+m_2\tfrac{1}{2}\theta^2+m_2\tfrac{1}{2}\phi^2)\end{aligned}$$

The linearised equations of motion are therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[m_1 l^2\dot{\theta} + m_2 l^2(\dot{\theta}+\dot{\phi})\right] + (m_1+m_2)gl\theta = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[m_2 l^2(\dot{\theta}+\dot{\phi})\right] + m_2 gl\phi = 0.$$

Let $\theta = \Theta\mathrm{e}^{\mathrm{i}\omega t}$, $\phi = \Phi\mathrm{e}^{\mathrm{i}\omega t}$, then

$$\begin{pmatrix} -\omega^2 l^2(m_1+m_2)+(m_1+m_2)gl & -\omega^2 l^2 m_2 \\ -\omega^2 l^2 m_2 & -\omega^2 l^2 m_2 + m_2 gl \end{pmatrix}\begin{pmatrix}\Theta\\ \Phi\end{pmatrix} = 0.$$

The vanishing of the determinant implies that

$$\begin{aligned}
& (m_1+m_2)(gl-\omega^2 l^2)m_2(gl-\omega^2l^2) = (\omega^2 l^2 m_2)^2 \\
\Rightarrow\quad & \sqrt{m_2(m_1+m_2)}(gl-\omega^2l^2) = \pm m_2\omega^2 l^2 \\
\Rightarrow\quad & \omega^2 l^2\left(\sqrt{1+\frac{m_1}{m_2}}\pm 1\right) = \sqrt{1+\frac{m_1}{m_2}}\,gl \\
\Rightarrow\quad & \omega^2 = \frac{g}{l}\frac{1}{1\pm 1/\sqrt{1+m_1/m_2}} = \frac{g}{l}\frac{1\mp 1/\sqrt{1+m_1/m_2}}{1-1/(1+m_1/m_2)} \\
& \quad = \frac{g}{l}\frac{1\mp 1/\sqrt{1+m_1/m_2}}{m_1/m_2/(1+m_1/m_2)} \\
& \quad = \frac{g}{l}\left(1+\frac{m_2}{m_1}\right)\left(1\mp 1\Big/\sqrt{1+\frac{m_1}{m_2}}\right).
\end{aligned}$$

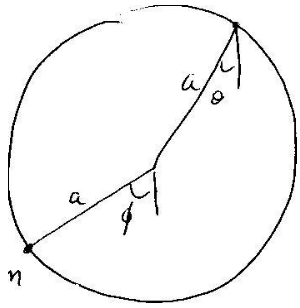
If $m_1 \gg m_2$, $\omega^2 \approx (g/l)(1\mp\sqrt{m_2/m_1})$ so both frequencies $\approx \sqrt{g/l}$ because the first mass swings without disturbance from the second.

If $m_2 \gg m_1$,

$$\omega^2 \approx \frac{g}{l}\frac{m_2}{m_1}\left[1\mp\left(1-\tfrac{1}{2}\frac{m_1}{m_2}\right)\right]$$

One frequency is now very high (the light mass m_1 on the taught string) and the other is $\approx \sqrt{g/2l}$ (mass on a string length $2l$).

5.



From the figure

$$T = \tfrac{1}{2}m(a\dot{\theta})^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + \tfrac{1}{2}m(a\dot{\theta}+a\dot{\phi})^2 = ma^2\dot{\theta}^2 + \tfrac{1}{2}ma^2(\dot{\theta}+\dot{\phi})^2$$

$$V = -mga\cos\theta - mg(a\cos\theta + a\cos\phi) = -2mga\cos\theta - mga\cos\phi$$

$$\simeq mga(\theta^2 + \tfrac{1}{2}\phi^2)$$

The linearized equations of motion are therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[2ma^2\dot{\theta} + ma^2(\dot{\theta}+\dot{\phi})\right] + 2mga\theta = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[ma^2(\dot{\theta}+\dot{\phi})\right] + mga\phi = 0$$

Putting in harmonic time dependence gives

$$-\omega^2(3\Theta + \Phi) + 2\frac{g}{a}\Theta = 0$$

$$-\omega^2(\Theta + \Phi) + \frac{g}{a}\Phi = 0$$

which implies that

$$0 = \left(2\frac{g}{a} - 3\omega^2\right)\left(\frac{g}{a} - \omega^2\right) - \omega^4 \quad \Rightarrow \quad 2\omega^4 - 5\omega^2\frac{g}{a} + 2\left(\frac{g}{a}\right)^2 = 0$$

Factorizing the quadratic in ω^2 we find that $\omega = \sqrt{g/2a}$ or $\omega = \sqrt{2g/a}$.

6.

$$T = \tfrac{1}{2}m(\dot{\mathbf{r}} + \omega\hat{\mathbf{k}}\times\mathbf{r})^2 = \tfrac{1}{2}\left[(\dot{x}-\omega y)^2 + (\dot{y}+\omega x)^2 + \dot{z}^2\right]$$

Hence

$$L = \tfrac{1}{2}m\left[(\dot{x}-\omega y)^2 + (\dot{y}+\omega x)^2 + \dot{z}^2\right] - \tfrac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)$$

and the eqns of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}m(\dot{x}-\omega y) - m(\dot{y}+\omega x)\omega + m\omega_x^2x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}m(\dot{y}+\omega x) + m(\dot{x}-\omega y)\omega + m\omega_y^2y = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}m\dot{z} + m\omega_z^2z = 0$$

The z motion decouples, so one normal frequency is ω_z

Letting $x = X\mathrm{e}^{\mathrm{i}\Omega t}$, $y = Y\mathrm{e}^{\mathrm{i}\Omega t}$, the eqns of motion yield

$$-\Omega^2X - \mathrm{i}\Omega\omega Y - \mathrm{i}\Omega\omega Y - \omega^2X + \omega_x^2X = 0$$

$$-\Omega^2Y + \mathrm{i}\Omega\omega X + \mathrm{i}\Omega\omega X - \omega^2Y + \omega_y^2Y = 0$$

$$\Rightarrow \quad \begin{vmatrix} -\Omega^2 - \omega^2 + \omega_x^2 & -2\mathrm{i}\Omega\omega \\ 2\mathrm{i}\Omega\omega & -\Omega^2 - \omega^2 + \omega_y^2 \end{vmatrix} = 0$$

Thus

$$0 = (-\Omega^2 - \omega^2 + \omega_x^2)(-\Omega^2 - \omega^2 + \omega_y^2) - 4\Omega^2\omega^2 = \Omega^4 + \Omega^2(2\omega^2 - \omega_x^2 - \omega_y^2 - 4\omega^2) + (\omega^2 - \omega_x^2)(\omega^2 - \omega_y^2)$$

From the usual formula for quadratics

$$\Omega^2 = \tfrac{1}{2}\left(2\omega^2 + \omega_x^2 + \omega_y^2 \pm \sqrt{(2\omega^2 + \omega_x^2 + \omega_y^2)^2 - 4(\omega^2 - \omega_x^2)(\omega^2 - \omega_y^2)}\right)$$

When $\omega_x > \omega > \omega_y$, we have $(\omega^2 - \omega_x^2)(\omega^2 - \omega_y^2) < 0$ so the radical is bigger than $2\omega^2 + \omega_x^2 + \omega_y^2$, so for one choice of sign $\Omega^2 < 0$, which implies that the motion is unstable.

7. $L = \frac{1}{2}m\left[\dot{r}^2 + (\omega r)^2\right] - \frac{1}{2}(mk/a)(r-a)^2$, so the eqn of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}m\dot{r} - m\omega^2 r + \frac{mk}{a}(r-a) = 0 \quad \Rightarrow \quad \ddot{r} + \left(\frac{k}{a} - \omega^2\right)r = k$$

The GS of this linear inhomogeneous equation of motion is

$$r = A\cos(\Omega t + \phi) + \frac{k}{\Omega^2} \qquad \Omega \equiv \sqrt{\frac{k}{a} - \omega^2}$$

We evaluate the arb consts A and ϕ from the given conditions at $t = 0$, and have

$$r = \left(a - \frac{k}{\Omega^2}\right)\cos\Omega t + \frac{k}{\Omega^2}$$

With f the reaction of the tube

$$fr = \text{torque} = \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\omega) = 2mr\dot{r}\omega \quad \Rightarrow \quad f = 2m\omega\dot{r}$$

$\dot{r}$ is maximum when $\Omega t = \pi/2$ with value $\dot{r}_{\max} = \Omega(a - k/\Omega^2)$ so

$$f_{\max} = 2ma\omega^3/\Omega.$$

8. In L the terms $\frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$ and $\frac{1}{2}(q_1^2 + q_2^2 + q_3^2)$ are manifestly invariant under rotations, so it remains only to show that $L' \equiv q_2q_3 + q_3q_1 + q_1q_2$ is invariant under rotations about $(1,1,1)$. We have $\delta\mathbf{q} = \delta\theta(1,1,1,) \times \mathbf{q} = \delta\theta(q_3 - q_2, q_1 - q_3, q_2 - q_1)$, so

$$\begin{aligned}
\delta L' &= \delta q_2 q_3 + q_2\delta q_3 + \delta q_3 q_1 + q_3\delta q_1 + \delta q_1 q_2 + q_1\delta q_2 \\
&= \delta q_1(q_3 + q_2) + \delta q_2(q_3 + q_1) + \delta q_3(q_1 + q_2) \\
&= \delta\theta\left[(q_3 - q_2)(q_3 + q_2) + (q_1 - q_3)(q_3 + q_1) + (q_2 - q_1)(q_1 + q_2)\right] \\
&= \delta\theta(q_3^2 - q_2^2 + q_1^2 - q_3^2 + q_2^2 - q_1^2) = 0
\end{aligned}$$

.

By Noether's theorem the constant of motion is

$$C = \frac{\delta\mathbf{q}}{\delta\theta} \cdot \frac{\partial L}{\partial\dot{\mathbf{q}}} = (q_3 - q_2, q_1 - q_3, q_2 - q_1) \cdot (\dot{q}_1, \dot{q}_2, \dot{q}_3) = \dot{q}_1(q_3 - q_2) + \dot{q}_2(q_1 - q_3) + \dot{q}_3(q_2 - q_1)$$

Explicitly calculating the derivative of C we find

$$\dot{C} = \ddot{q}_1(q_3 - q_2) + \ddot{q}_2(q_1 - q_3) + \ddot{q}_3(q_2 - q_1) + \dot{q}_1(\dot{q}_3 - \dot{q}_2) + \dot{q}_2(\dot{q}_1 - \dot{q}_3) + \dot{q}_3(\dot{q}_2 - \dot{q}_1)$$

But

$$\begin{aligned}
\ddot{q}_1 + q_1 + \alpha(q_3 + q_2) &= 0 \\
\ddot{q}_2 + q_2 + \alpha(q_1 + q_3) &= 0 \\
\ddot{q}_3 + q_3 + \alpha(q_2 + q_1) &= 0
\end{aligned}$$

so

$$-\dot{C} = [q_1 + \alpha(q_3 + q_2)](q_3 - q_2) + [q_2 + \alpha(q_1 + q_3)](q_1 - q_3) + [q_3 + \alpha(q_2 + q_1)](q_2 - q_1) = 0.$$

9. In a spherical potential the angular momentum per unit mass, $\mathbf{r}\times\dot{\mathbf{r}}$, is conserved, so taking the derivative of K we find

$$0=\dot{K}=\ddot{\mathbf{r}}\times(\mathbf{r}\times\dot{\mathbf{r}})+\alpha'\frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r}\mathbf{r}+\alpha\dot{\mathbf{r}},$$

where we've used $\dot{r}=\mathbf{r}\cdot\dot{\mathbf{r}}/r$ and $\alpha'\equiv \mathrm{d}\alpha/\mathrm{d}r$. Using $\ddot{\mathbf{r}}=-\nabla V=-V'\hat{\mathbf{r}}$ we have

$$\begin{aligned}0&=-\frac{V'}{r}\mathbf{r}\times(\mathbf{r}\times\dot{\mathbf{r}})+\frac{\alpha'}{r}(\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}+\alpha\dot{\mathbf{r}}\\&=-\frac{V'}{r}[(\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}-r^2\dot{\mathbf{r}}]+\frac{\alpha'}{r}(\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}+\alpha\dot{\mathbf{r}}\end{aligned}$$

We now use tensor notation to extract a factor of the velocity:

$$0=\dot{r}_i\left[rV'\delta_{ij}-\frac{V'}{r}r_ir_j+\frac{\alpha'}{r}r_ir_j+\alpha\delta_{ij}\right].$$

As initial conditions we get to choose $\dot{\mathbf{r}}$, so if this equation is to hold along any orbit, the tensor multiplying $\dot{\mathbf{r}}$ must vanish. We also get to choose $\mathbf{r}$, so if the tensor is to vanish for any orbit, the coefficients of δ_{ij} and r_ir_j must separately vanish. It follows that

$$rV'=-\alpha\quad\text{and}\quad V'=\alpha'\quad\Rightarrow\quad\frac{\mathrm{d}\alpha}{\alpha}=-\frac{\mathrm{d}r}{r}\quad\Rightarrow\quad\alpha=A/r\quad\Rightarrow\quad V=-A/r\quad(A\text{a const.})$$

Similarly, taking the derivative of Q_{ij} we find

$$\begin{aligned}0=\dot{Q}_{ij}&=\ddot{r}_i\dot{r}_j+\dot{r}_i\ddot{r}_j+\frac{\beta'}{r}\mathbf{r}\cdot\dot{\mathbf{r}}\,r_ir_j+\beta(\dot{r}_ir_j+r_i\dot{r}_j)\\&=-\frac{V'}{r}(r_i\dot{r}_j+\dot{r}_ir_j)+\frac{\beta'}{r}\mathbf{r}\cdot\dot{\mathbf{r}}\,r_ir_j+\beta(\dot{r}_ir_j+r_i\dot{r}_j)\end{aligned}$$

The argumentation used in the first part now implies that

$$0=\beta-\frac{V'}{r}\quad\text{and}\quad 0=\beta'$$

from which it follows that $V=\frac{1}{2}\beta r^2$, where β is a constant.