Classical Mechanics I

1. A chain of length l is hung from two points that are at the same level but are distance $s < l$ apart. The chain adopts that curve $z(x)$ (a **catenary**) which minimizes it potential energy $W[z(x)]$. By minimizing the chain's potential energy subject to its length being l, show that z satisfies

$$(z-\lambda)\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 - 1 = 0,$$

where λ is a Lagrange multiplier.

Solve for $z(x)$. [Hint: define $u \equiv \mathrm{d}z/\mathrm{d}x$ and show that

$$\frac{u\,\mathrm{d}u}{1+u^2} = \frac{\mathrm{d}z}{z-\lambda} \quad \Big].$$

2. Write down the Lagrangian for the motion of a particle of mass m in a potential $V(r,\phi)$ when referred to planar polar coordinates (r,ϕ). Hence show that the equations of motion are

$$m\ddot{r} - mr\dot{\phi}^2 = -\frac{\partial V}{\partial r} \qquad mr\ddot{\phi} + 2m\dot{r}\dot{\phi} = -\frac{1}{r}\frac{\partial V}{\partial \phi}.$$

3. A particle of mass m moves in a spherically-symmetric potential $V(r)$. Show that the motion is confined to a plane.

Obtain the Lagrangian for motion in this plane in terms of the variables $u \equiv 1/r$ and the angle ϕ. Show that if $V(r) = -\alpha/r$ one has

$$u(\phi) = A\cos(\phi - \phi_0) + B,$$

where A, B and ϕ_0 are arbitrary constants. Show that the orbit is an ellipse if $B > A$ and a parabola or hyperbola otherwise.

4. Use a Lagrangian to show that when referred to spherical polar coordinates, the equations of motion of a particle in a gravitational potential $V(\mathbf{x})$ are

$$\begin{aligned}
0 &= \ddot{r} - r\big(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\big) + \frac{\partial V}{\partial r} \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\big(r^2\dot{\theta}\big) - r^2\dot{\phi}^2\sin\theta\cos\theta + \frac{\partial V}{\partial \theta} \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\big(r^2\sin^2\theta\dot{\phi}\big) + \frac{\partial V}{\partial \phi}.
\end{aligned}$$

In the case in which $V = V(r)$ is spherically symmetric, show that

$$r^2\sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}$$

is a conserved quantity and interpret this result physically.

5. The bottom spike of an axisymmetric top of mass m lies distance a below the top's centre of gravity. Show that when the top is spinning with its spike in contact with a rough floor, the system's Lagrangian is

$$L = \tfrac{1}{2}I_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - mga\cos\theta,$$

where (θ, ϕ, ψ) are Euler angles relative to a vertical $\mathbf{k}$ axis and I_3 is the principal moment of inertia about the top's symmetry axis. Show that the top can precess steadily at fixed inclination to the vertical only if θ satisfies

$$0 = mga + (I_1 - I_3)\dot{\phi}^2\cos\theta - I_3\dot{\phi}\dot{\psi}.$$

6. A particle of mass m_1 hangs by a light string of length l from a rigid support, and a second mass, m_2, hangs by an identical string from m_1. The angles with the vertical of the strings supporting m_1 and m_2 are θ and ϕ, respectively. Write down the Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi})$ of the system. Hence show that the frequencies of the two normal modes of oscillation about equilibrium are $\omega_\pm$, where

$$\omega_\pm^2 = \frac{g}{l}\frac{m_1 + m_2}{m_1}\left[1 \pm \sqrt{\frac{m_2}{m_1 + m_2}}\right].$$

Describe the motion in each of the normal modes in the cases (a) $m_1 \gg m_2$, and (b) $m_2 \gg m_1$.

7. A circular hoop of mass m and radius a hangs from a point on its circumference and is free to oscillate in its own plane. A bead of mass m can slide without friction around the hoop. Choose a set of generalized coordinates and write down the Lagrangian for the system. Show that the natural frequencies for small oscillations about equilibrium are $\omega_1 = \sqrt{2g/a}$ and $\omega_2 = \sqrt{g/2a}$.

8. The (x, y, z) frame of reference rotates with angular speed $\boldsymbol{\omega} = \omega\mathbf{k}$. A particle of mass m moves in the potential

$$V(x, y, z) = \tfrac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$

By solving for the frequencies of the particle's normal modes about the equilibrium $\mathbf{x} = 0$, show that the motion is unstable if $\omega_x < \omega < \omega_y$.

9. A particle of mass m slides inside a smooth straight tube OA to which it is connected at point O by a light spring of natural length a and spring constant mk/a. The system rotates in a horizontal plane with constant angular velocity ω about a fixed vertical axis through O. Determine the distance r of the particle from O at time t for the case when $\omega^2 < k/a$, if $r = a$ and $\dot{r} = 0$ at $t = 0$. Show also for this case that the maximum value of the reaction of the tube on the particle is $2ma\omega^3/b$, where $b^2 \equiv (k/a - \omega^2)$.

10. What is meant by the terms *symmetry principle* and *conservation law* as used in classical dynamics? Give simple examples to illustrate the symmetries underlying the conservation of linear and angular momentum.

A system with three degrees of freedom described by coordinates q_1, q_2, q_3 has Lagrangian

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \tfrac{1}{2}(q_1^2 + q_2^2 + q_3^2) - \alpha(q_2 q_3 + q_3 q_1 + q_1 q_2),$$

where $0 < \alpha < \frac{1}{2}$. Show that L is invariant under infinitesimal rotations about the $(1, 1, 1)$ axis in q-space, and hence find a constant of motion other than the total energy. Verify from the equations of motion that it is indeed constant.

11. A particle with position coordinates $\mathbf{r}$ moves in a central potential $V(r)$. By considering the quantity $(\mathbf{r} \times \dot{\mathbf{r}})$ show that the orbit of the particle lies in a fixed plane.

Find all potential functions $V(r)$ and corresponding functions $\alpha(r)$ for which the vector

$$\mathbf{K} = \dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}}) + \alpha(r)\mathbf{r}$$

is conserved.

Find also the potentials $V(r)$ and functions $\beta(r)$ for which the components of the matrix

$$Q_{ij} \equiv \dot{r}_i \dot{r}_j + \beta(r) r_i r_j$$

are constants of the motion, where r_i, $\dot{r}_i$ $(i = 1, 2, 3)$ are the components of position and velocity of the particle along any three independent fixed axes.

Classical Mechanics II

1. Show that if the Hamiltonian is independent of a generalized coordinate q_0, then the conjugate momentum p_0 is a constant of motion. Such coordinates are called **cyclic coordinates**. Give two examples of physical systems that have a cyclic coordinate.

2. Show that in spherical polar coordinates the Hamiltonian of a particle of mass m moving in a potential $V(\mathbf{x})$ is

$$H = \frac{1}{2m}\Big(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\Big) + V(\mathbf{x}).$$

Show that $p_\phi =$ constant when $\partial V/\partial\phi \equiv 0$ and interpret this result physically.

Show that $[H, K] = 0$ where $K \equiv p_\theta^2 + \dfrac{p_\phi^2}{\sin^2\theta}$. By expressing K as a function of $\dot\theta$ and $\dot\phi$ interpret this result physically.

Consider circular motion with angular momentum h in a spherical potential $V(r)$. Evaluate $p_\theta(\theta)$ when the orbit's plane is inclined by ψ to the equatorial plane. Show that $p_\theta = 0$ when $\sin\theta = \pm\cos\psi$ and iterpret this result physically.

3. The bottom spike of an axisymmetric top of mass m lies distance a below the top's centre of gravity. Show that when the top is spinning with its spike in contact with a rough floor, the system's Hamiltonian is

$$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + mga\cos\theta.$$

where (θ, ϕ, ψ) are Euler angles relative to a vertical $\mathbf{k}$ axis and I_3 is the principal moment of inertia about the top's symmetry axis. Identify two constants of the motion in addition to H.

Show that the top will precess steadily at fixed inclination to the vertical provided θ satisfies

$$0 = mga + \frac{(p_\phi - p_\psi\cos\theta)(p_\phi\cos\theta - p_\psi)}{I_1\sin^4\theta}.$$

4. Oblate spheroidal coordinates (u, v, ϕ) are related to regular cylindrical polars (R, z, ϕ) by

$$R = \Delta\cosh u\cos v \quad ; \quad z = \Delta\sinh u\sin v.$$

Show that in these coordinates momenta of a particle of mass m are

$$\begin{aligned} p_u &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot u, \\ p_v &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot v, \\ p_\phi &= m\Delta^2\cosh^2 u\cos^2 v\dot\phi. \end{aligned}$$

Hence show that the Hamiltonian for motion in a potential $\Phi(u, v)$ is

$$H = \frac{p_u^2 + p_v^2}{2m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_\phi^2}{2m\Delta^2\cosh^2 u\cos^2 v} + \Phi.$$

Show that $[H, p_\phi] = 0$ and hence that p_ϕ is a constant of motion. Identify it physically.

5. A particle of mass m and charge Q moves in the equatorial plane of a magnetic dipole. Given that the dipole has vector potential

$$\mathbf{A} = \frac{\mu_0 \mu \sin\theta}{4\pi r^2}\,\mathbf{e}_\phi,$$

evaluate the Hamiltonian $H(p_r, p_\phi, r, \phi)$ of the system.

The particle approaches the dipole from infinity at speed v and impact parameter b. Show that p_ϕ and the particle's speed are constants of motion.

Show further that for $Q\mu > 0$ the distance of closest approach to the dipole is

$$D = \tfrac{1}{2}\begin{cases} \sqrt{b^2 - a^2} + b & \text{for } b > a \\ \sqrt{b^2 + a^2} - b & \text{for } b < a \end{cases} \quad \text{where} \quad a^2 \equiv \frac{\mu_0 Q \mu}{\pi m v}.$$

6. A point charge q is placed at the origin in the magnetic field generated by a spatially confined current distribution. Given that

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$$

and $\mathbf{B} = \nabla \times \mathbf{A}$ with $\nabla \cdot \mathbf{A} = 0$, show that the field's momentum

$$\mathbf{P} \equiv \epsilon_0 \int \mathbf{E} \times \mathbf{B}\,\mathrm{d}^3\mathbf{x} = q\mathbf{A}(0).$$

Use this result to interpret the formula for the canonical momentum of a charged particle in an e.m. field. [Hint: write $\mathbf{E} = -(q/4\pi\epsilon_0)\nabla r^{-1}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, expand the vector triple product and integrate each of the resulting terms by parts so as to exploit in one $\nabla \cdot \mathbf{A} = 0$ and in the other $\nabla^2 r^{-1} = -4\pi\delta^3(\mathbf{r})$. The tensor form of Gauss's theorem states that $\int \mathrm{d}^3\mathbf{x}\,\nabla_i \mathbf{T} = \oint \mathrm{d}^2 S_i\,\mathbf{T}$ no matter how many indices the tensor $\mathbf{T}$ may carry.]

7. For each convex function $f(x)$, i.e. for each $f(x)$ for which $f''(x) > 0$, define $F(x, p)$ to be the function of two variables

$$F(x, p) \equiv xp - f(x).$$

Show that for each fixed p, $F(x, p)$ has a unique maximum with respect to x when $f'(x) = p$. Let this maximum occur at x_p. We define the Legendre transform of f to be

$$\overline{f}(p) \equiv F(x_p, p).$$

Show that the Legendre transform $\overline{\overline{f}}(q)$ of $\overline{f}(p)$ is $\overline{\overline{f}}(q) = f(q)$. (In other words on applying the transform twice you recover your original function.)

[Hint: first show that $qp - \overline{f}(p)$ achieves its maximum w.r.t. p when $x_p = q$.]

8. Show that the generating function of the form $S(\mathbf{P}, \mathbf{x})$ which generates the Gallilean transformation between frames in relative motion at velocity $\mathbf{V}$ is

$$S = \mathbf{P} \cdot \mathbf{x} + \mathbf{V} \cdot (m\mathbf{x} - t\mathbf{P}).$$

9. A point transformation is specified by n functions $Q_j(\mathbf{q})$ of the old coordinates $\mathbf{q}$. Show that any point transformation is canonical by evaluating $[Q_i, Q_j]$, $[P_i, P_j]$, etc., where $\mathbf{P} \equiv \partial L/\partial \dot{\mathbf{Q}}$, with L the Lagrangian. [Hint: you may find it useful to prove first that $\dot{Q}_i = (\partial Q_i/\partial q_j)\dot{q}_j$ and $P_i = p_j(\partial q_j/\partial Q_i)$.]

10. Show that for a harmonic oscillator of frequency ω the Hamilton-Jacobi equation reads

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}\right)^2 + m^2\omega^2x^2 = 2mE.$$

Identify a new momentum P which allows S to be written

$$S(P,x) = (\theta + \tfrac{1}{2}\sin 2\theta)P \quad \text{where} \quad \theta(P,x) \equiv \arcsin\left(\sqrt{\frac{m\omega}{2P}}x\right).$$

Hence show that the action-angle coordinates of this system may be taken to be

$$\begin{aligned} P &\equiv \frac{1}{2m\omega}(p^2 + m^2\omega^2x^2), \\ Q &\equiv \arctan(m\omega x/p). \end{aligned}$$

(Notice that according to quantum mechanics $P/\hbar = (n + \frac{1}{2})$ takes half-integral values. The 'old quantum theory' was founded on assigning such special values to action variables divided by $\hbar$.)

11. Show that when the potential of the Problem 4 is of the form

$$\Phi(u,v) = \frac{U(u) - V(v)}{\cosh^2 u - \cos^2 v}, \qquad (\dagger)$$

the Hamilton-Jacobi equation separates. Hence show that in the case $p_\phi = 0$ the other momenta are related to the coordinates by

$$\begin{aligned} p_u &= \pm\Delta\sqrt{2m[E\cosh^2 u - I - U(u)]} \\ p_v &= \pm\Delta\sqrt{2m[-E\cos^2 v + I + V(v)]}, \end{aligned}$$

where I is a constant of separation. Express I as a function of position in phase space. (Potentials of the form (†) are called **Stäckel potentials** after P. Stäckel, who demonstrated that ellipsoidal coordinates provide the most general coordinate system in which one can separate the Hamilton-Jacobi equation of a particle moving in $\Phi(\mathbf{x})$.)