Classical Mechanics: Additional off-syllabus problems

1. *A chain of length l is hung from two points that are at the same level but are distance $s < l$ apart. The chain adopts that curve $z(x)$ (a* **catenary***) which minimizes it potential energy $W[z(x)]$. By minimizing the chain's potential energy subject to its length being l, show that z satisfies*

$$(z-\lambda)\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 - 1 = 0,$$

where λ is a Lagrange multiplier.

Solve for $z(x)$. [Hint: define $u \equiv \mathrm{d}z/\mathrm{d}x$ and show that

$$\frac{u\,\mathrm{d}u}{1+u^2} = \frac{\mathrm{d}z}{z-\lambda} \quad \Big].$$

2. *The bottom spike of an axisymmetric top of mass m lies distance a below the top's centre of gravity. Show that when the top is spinning with its spike in contact with a rough floor, the system's Lagrangian is*

$$L = \tfrac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \tfrac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - mga\cos\theta,$$

where (θ,ϕ,ψ) are Euler angles relative to a vertical $\mathbf{k}$ axis and I_3 is the principal moment of inertia about the top's symmetry axis. Show that the top can precess steadily at fixed inclination to the vertical only if θ satisfies

$$0 = mga + (I_1 - I_3)\dot\phi^2\cos\theta - I_3\dot\phi\dot\psi.$$

The top precesses steadily iff $\dot\theta = 0$. The θ equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_1\theta_1) - I_1\dot\phi^2\sin\theta\cos\theta + I_3(\dot\phi\cos\theta + \dot\psi)\dot\phi\sin\theta - mga\sin\theta = 0,$$

so $\dot\theta = 0$ at all times requires either $\sin\theta = 0$ or

$$0 = mga + (I_1 - I_3)\dot\phi^2\cos\theta - \dot\phi\dot\psi I_3.$$

3. *The bottom spike of an axisymmetric top of mass m lies distance a below the top's centre of gravity. Show that when the top is spinning with its spike in contact with a rough floor, the system's Hamiltonian is*

$$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + mga\cos\theta.$$

where (θ,ϕ,ψ) are Euler angles relative to a vertical $\mathbf{k}$ axis and I_3 is the principal moment of inertia about the top's symmetry axis. Identify two constants of the motion in addition to H.

Show that the top will precess steadily at fixed inclination to the vertical provided θ satisfies

$$0 = mga + \frac{(p_\phi - p_\psi\cos\theta)(p_\phi\cos\theta - p_\psi)}{I_1\sin^4\theta}.$$

The Lagrangian for this system is

$$L = \tfrac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \tfrac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - mga\cos\theta$$

$$p_\theta = I_1\dot\theta,\ p_\phi = I_1\sin^2\theta\dot\phi + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta,\ p_\psi = I_3(\dot\phi\cos\theta + \dot\psi) \quad \Rightarrow \quad I_1\sin^2\theta\dot\phi = p_\phi - \cos\theta p_\psi$$

$$\begin{aligned} H &= I_1\dot\theta^2 + I_1\sin^2\theta\dot\phi^2 + I_3[(\dot\phi\cos\theta + \dot\psi)\cos\theta\dot\phi \\ &\quad + (\dot\phi\cos\theta + \dot\psi)\dot\psi] - \tfrac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) - \tfrac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 + mga\cos\theta \\ &= \tfrac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \tfrac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 + mga\cos\theta \\ &= \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + mga\cos\theta \end{aligned}$$

H doesn't depend on either ϕ or ψ, so p_ϕ and p_ψ are constants of the motion.

For steady precession we require

$$0 = \dot\theta = \frac{\partial H}{\partial p_\theta} \quad \Rightarrow \quad p_\theta = 0$$

We also require

$$0 = \dot p_\theta = -\frac{\partial H}{\partial \theta} = -\frac{(p_\phi - p_\psi\cos\theta)^2}{I_1\sin^3\theta}\cos\theta + \frac{(p_\phi - p_\psi\cos\theta)}{I_1\sin^2\theta}p_\psi\sin\theta - mga\sin\theta$$

$$\Rightarrow \quad 0 = mga + \frac{(p_\phi - p_\psi\cos\theta)}{I_1\sin^4\theta}(p_\phi\cos\theta - p_\psi\cos^2\theta - p_\psi\sin^2\theta)$$

4. Show that for a harmonic oscillator of frequency ω the Hamilton-Jacobi equation reads

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}\right)^2 + m^2\omega^2x^2 = 2mE.$$

Identify a new momentum P which allows S to be written

$$S(P,x) = (\theta + \tfrac{1}{2}\sin 2\theta)P \quad \text{where} \quad \theta(P,x) \equiv \arcsin\left(\sqrt{\frac{m\omega}{2P}}x\right).$$

Hence show that the action-angle coordinates of this system may be taken to be

$$\begin{aligned} P &\equiv \frac{1}{2m\omega}(p^2 + m^2\omega^2x^2), \\ Q &\equiv \arctan(m\omega x/p). \end{aligned}$$

(Notice that according to quantum mechanics $P/\hbar = (n + \frac{1}{2})$ takes half-integral values. The 'old quantum theory' was founded on assigning such special values to action variables divided by $\hbar$.)

5. Show that when the potential of Problem 5 on problem set II is of the form

$$\Phi(u,v) = \frac{U(u) - V(v)}{\cosh^2 u - \cos^2 v}, \qquad (\dagger)$$

the Hamilton-Jacobi equation separates. Hence show that in the case $p_\phi = 0$ the other momenta are related to the coordinates by

$$\begin{aligned} p_u &= \pm\Delta\sqrt{2m[E\cosh^2 u - I - U(u)]} \\ p_v &= \pm\Delta\sqrt{2m[-E\cos^2 v + I + V(v)]}, \end{aligned}$$

where I is a constant of separation. Express I as a function of position in phase space. [Potentials of the form (†) are called **Stäckel potentials** after P. Stäckel, who demonstrated that ellipsoidal coordinates provide the most general coordinate system in which one can separate the Hamilton-Jacobi equation of a particle moving in $\Phi(\mathbf{x})$.]