Classical Mechanics II

1. Show that if the Hamiltonian is independent of a generalized coordinate q_0, then the conjugate momentum p_0 is a constant of motion. Such coordinates are called **cyclic coordinates**. Give two examples of physical systems that have a cyclic coordinate.

2. A dynamical system has generalized co-ordinates q_i and generalized momenta p_i.

Verify the following properties of the Poisson brackets:

$$[q_i, q_j] = [p_i, p_j] = 0; \qquad [q_i, p_j] = \delta_{ij}.$$

If $\mathbf{p}$ is the momentum conjugate to a position vector $\mathbf{r}$, and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, evaluate the Poisson brackets $[L_x, L_y]$, $[L_y, L_x]$ and $[L_x, L_x]$. Comment on their significance.

The Lagrangian of a particle of mass m and charge e in a uniform magnetic field $\mathbf{B}$ and an electrostatic potential ϕ is

$$L = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + \tfrac{1}{2}e\dot{\mathbf{r}} \cdot (\mathbf{B} \times \mathbf{r}) - e\phi.$$

Derive the corresponding Hamiltonian and verify that the rate of change of $m\dot{\mathbf{r}}$ equals the Lorentz force. Show that the momentum component along $\mathbf{B}$ and the sum of the squares of the two other momentum components are all constants of motion. Find another constant of motion associated with time translation symmetry.

3. Let p and q be canonically conjugate coordinates, and let $f(p, q)$ and $g(p, q)$ be functions on phase space. Define the Poisson bracket $[f, g]$. Let $H(p, q)$ be the Hamiltonian that governs the system's dynamics. Write down the equations of motion of p and q in terms of H and the Poisson bracket.

In a galaxy, the density of stars in phase space is $f(\mathbf{p}, \mathbf{q}, t)$, where $\mathbf{p}$ and $\mathbf{q}$ each have three components. When evaluated at the location $\big(\mathbf{p}(t), \mathbf{q}(t)\big)$ of any given star, f is time-independent. Show that f consequently satisfies

$$\frac{\partial f}{\partial t} = [H, f],$$

where H is the Hamiltonian that governs the motion of every star.

Consider motion in a circular, razor-thin galaxy in which the potential energy of any star is given by the function $V(R)$, where R is a radical coordinate. Express H in terms of plane polar coordinates R, ϕ and their conjugate momenta, with the origin coinciding with the galaxy's centre. Hence, or otherwise, show that in this system f satisfies the equation

$$\frac{\partial f}{\partial t} + \frac{p_R}{m}\frac{\partial f}{\partial R} + \frac{p_\phi}{mR^2}\frac{\partial f}{\partial \phi} - \left(\frac{\partial V}{\partial R} - \frac{{p_\phi}^2}{mR^3}\right)\frac{\partial f}{\partial p_R} = 0,$$

where m is are the mass of the star.

4. Show that in spherical polar coordinates the Hamiltonian of a particle of mass m moving in a potential $V(\mathbf{x})$ is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V(\mathbf{x}).$$

Show that $p_\phi =$ constant when $\partial V/\partial\phi \equiv 0$ and interpret this result physically.

Given that V depends only on r, show that $[H, K] = 0$ where $K \equiv p_\theta^2 + \dfrac{p_\phi^2}{\sin^2\theta}$. By expressing K as a function of $\dot\theta$ and $\dot\phi$ interpret this result physically.

Consider circular motion with angular momentum h in a spherical potential $V(r)$. Evaluate $p_\theta(\theta)$ when the orbit's plane is inclined by ψ to the equatorial plane. Show that $p_\theta = 0$ when $\sin\theta = \pm\cos\psi$ and iterpret this result physically.

5. Oblate spheroidal coordinates (u, v, ϕ) are related to regular cylindrical polars (R, z, ϕ) by

$$R = \Delta \cosh u \cos v \quad ; \quad z = \Delta \sinh u \sin v.$$

Show that in these coordinates momenta of a particle of mass m are

$$\begin{aligned} p_u &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot{u}, \\ p_v &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot{v}, \\ p_\phi &= m\Delta^2 \cosh^2 u \cos^2 v\dot{\phi}. \end{aligned}$$

Hence show that the Hamiltonian for motion in a potential $\Phi(u, v)$ is

$$H = \frac{p_u^2 + p_v^2}{2m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_\phi^2}{2m\Delta^2 \cosh^2 u \cos^2 v} + \Phi.$$

Show that $[H, p_\phi] = 0$ and hence that p_ϕ is a constant of motion. Identify it physically.

6. A particle of mass m and charge Q moves in the equatorial plane $\theta = \pi/2$ of a magnetic dipole. Given that the dipole has vector potential

$$\mathbf{A} = \frac{\mu \sin\theta}{4\pi r^2}\mathbf{e}_\phi,$$

evaluate the Hamiltonian $H(p_r, p_\phi, r, \phi)$ of the system.

The particle approaches the dipole from infinity at speed v and impact parameter b. Show that p_ϕ and the particle's speed are constants of motion.

Show further that for $Q\mu > 0$ the distance of closest approach to the dipole is

$$D = \tfrac{1}{2}\begin{cases} b - \sqrt{b^2 - a^2} & \text{for } \dot{\phi} > 0 \\ b + \sqrt{b^2 + a^2} & \text{for } \dot{\phi} < 0 \end{cases} \quad \text{where} \quad a^2 \equiv \frac{\mu Q}{\pi m v}.$$

7. A point charge q is placed at the origin in the magnetic field generated by a spatially confined current distribution. Given that

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$$

and $\mathbf{B} = \nabla \times \mathbf{A}$ with $\nabla \cdot \mathbf{A} = 0$, show that the field's momentum

$$\mathbf{P} \equiv \epsilon_0 \int \mathbf{E} \times \mathbf{B}\, \mathrm{d}^3\mathbf{x} = q\mathbf{A}(0).$$

Use this result to interpret the formula for the canonical momentum of a charged particle in an e.m. field. [Hint: write $\mathbf{E} = -(q/4\pi\epsilon_0)\nabla r^{-1}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, expand the vector triple product and integrate each of the resulting terms by parts so as to exploit in one $\nabla \cdot \mathbf{A} = 0$ and in the other $\nabla^2 r^{-1} = -4\pi\delta^3(\mathbf{r})$. The tensor form of Gauss's theorem states that $\int \mathrm{d}^3\mathbf{x}\, \nabla_i \mathbf{T} = \oint \mathrm{d}^2 S_i\, \mathbf{T}$ no matter how many indices the tensor $\mathbf{T}$ may carry.]

8. For each convex function $f(x)$, i.e. for each $f(x)$ for which $f''(x) > 0$, define $F(x, p)$ to be the function of two variables

$$F(x, p) \equiv xp - f(x).$$

Show that for each fixed p, $F(x, p)$ has a unique maximum with respect to x when $f'(x) = p$. Let this maximum occur at x_p. We define the Legendre transform of f to be

$$\overline{f}(p) \equiv F(x_p, p).$$

Show that the Legendre transform $\overline{\overline{f}}(q)$ of $\overline{f}(p)$ is $\overline{\overline{f}}(q) = f(q)$. (In other words on applying the transform twice you recover your original function.)

[Hint: first show that $qp - \overline{f}(p)$ achieves its maximum w.r.t. p when $x_p = q$.]

9. Show that the generating function of the form $S(\mathbf{P}, \mathbf{x})$ which generates the Gallilean transformation between frames in relative motion at velocity $\mathbf{V}$ is

$$S = \mathbf{P} \cdot \mathbf{x} + \mathbf{V} \cdot (m\mathbf{x} - t\mathbf{P}).$$

10. A point transformation is specified by n functions $Q_j(\mathbf{q})$ of the old coordinates $\mathbf{q}$. Show that any point transformation is canonical by evaluating $[Q_i, Q_j]$, $[P_i, P_j]$, etc., where $\mathbf{P} \equiv \partial L/\partial\dot{\mathbf{Q}}$, with L the Lagrangian. [Hint: you may find it useful to prove first that $\dot{Q}_i = (\partial Q_i/\partial q_j)\dot{q}_j$ and $P_i = p_j(\partial q_j/\partial Q_i)$.]