Classical Mechanics I

1. Write down the Lagrangian for the motion of a particle of mass m in a potential $V(r,\phi)$ when referred to planar polar coordinates (r,ϕ). Hence show that the equations of motion are

$$m\ddot{r} - mr\dot{\phi}^2 = -\frac{\partial V}{\partial r} \qquad mr\ddot{\phi} + 2m\dot{r}\dot{\phi} = -\frac{1}{r}\frac{\partial V}{\partial \phi}.$$

Hence derive the principle of conservation of angular momentum in the plane, and obtain the usual formula v^2/r for centripetal acceleration.

2. A particle of mass m moves in a spherically-symmetric potential $V(r)$. Show that the motion is confined to a plane.

Obtain the Lagrangian for motion in this plane in terms of the variables $u \equiv 1/r$ and the angle ϕ. Show that if $V(r) = -\alpha/r$ one has

$$u(\phi) = A\cos(\phi - \phi_0) + B,$$

where A, B and ϕ_0 are arbitrary constants. Show that the orbit is an ellipse if $B > A$ and a parabola or hyperbola otherwise.

3. Use a Lagrangian to show that when referred to spherical polar coordinates, the equations of motion of a particle in a gravitational potential $\Phi(\mathbf{x})$ are

$$\begin{aligned}
0 &= \ddot{r} - r\big(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\big) + \frac{\partial \Phi}{\partial r}\\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\big(r^2\dot{\theta}\big) - r^2\dot{\phi}^2\sin\theta\cos\theta + \frac{\partial \Phi}{\partial \theta}\\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\big(r^2\sin^2\theta\dot{\phi}\big) + \frac{\partial \Phi}{\partial \phi}.
\end{aligned}$$

In the case in which $\Phi = \Phi(r)$ is spherically symmetric, show that

$$r^2\sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}$$

is a conserved quantity and interpret this result physically.

4. A particle of mass m_1 hangs by a light string of length l from a rigid support, and a second mass, m_2, hangs by an identical string from m_1. The angles with the vertical of the strings supporting m_1 and m_2 are θ and ϕ, respectively. Write down the Lagrangian $L(\theta,\phi,\dot{\theta},\dot{\phi})$ of the system. Hence show that the frequencies of the two normal modes of oscillation about equilibrium are $\omega_\pm$, where

$$\omega_\pm^2 = \frac{g}{l}\frac{m_1+m_2}{m_1}\left[1 \pm \sqrt{\frac{m_2}{m_1+m_2}}\right].$$

Describe the motion in each of the normal modes in the cases (a) $m_1 \gg m_2$, and (b) $m_2 \gg m_1$.

5. A circular hoop of mass m and radius a hangs from a point on its circumference and is free to oscillate in its own plane. A bead of mass m can slide without friction around the hoop. Choose a set of generalized coordinates and write down the Lagrangian for the system. Show that the natural frequencies for small oscillations about equilibrium are $\omega_1 = \sqrt{2g/a}$ and $\omega_2 = \sqrt{g/2a}$.

6. The (x,y,z) frame of reference rotates with angular speed $\boldsymbol{\omega} = \omega\mathbf{k}$. A particle of mass m moves in the potential

$$V(x,y,z) = \tfrac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2).$$

By solving for the frequencies of the particle's normal modes about the equilibrium $\mathbf{x} = 0$, show that the motion is unstable if $\omega_x < \omega < \omega_y$.

7. A particle of mass m slides inside a smooth straight tube OA to which it is connected at point O by a light spring of natural length a and spring constant mk/a. The system rotates in a horizontal plane with constant angular velocity ω about a fixed vertical axis through O. Determine the distance r of the particle from O at time t for the case when $\omega^2 < k/a$, if $r = a$ and $\dot{r} = 0$ at $t = 0$. Show also for this case that the maximum value of the reaction of the tube on the particle is $2ma\omega^3/b$, where $b^2 \equiv (k/a - \omega^2)$.

8. What is meant by the terms *symmetry principle* and *conservation law* as used in classical dynamics? Give simple examples to illustrate the symmetries underlying the conservation of linear and angular momentum.

A system with three degrees of freedom described by coordinates q_1, q_2, q_3 has Lagrangian

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \tfrac{1}{2}(q_1^2 + q_2^2 + q_3^2) - \alpha(q_2 q_3 + q_3 q_1 + q_1 q_2),$$

where $0 < \alpha < \frac{1}{2}$. Show that L is invariant under infinitesimal rotations about the $(1, 1, 1)$ axis in q-space, and hence find a constant of motion other than the total energy. Verify from the equations of motion that it is indeed constant.

9. A particle with position coordinates $\mathbf{r}$ moves in a central potential $V(r)$. Find all potential functions $V(r)$ and corresponding functions $\alpha(r)$ for which the vector

$$\mathbf{K} = \dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}}) + \alpha(r)\mathbf{r}$$

is conserved.

Find also the potentials $V(r)$ and functions $\beta(r)$ for which the components of the matrix

$$Q_{ij} \equiv \dot{r}_i \dot{r}_j + \beta(r) r_i r_j$$

are constants of the motion, where r_i, $\dot{r}_i$ $(i = 1, 2, 3)$ are the components of position and velocity of the particle along any three independent fixed axes.