Classical Fields II: Solutions

1.

$$\delta s = 0 \quad \Leftrightarrow \quad 0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{g_{\kappa\nu}\mathrm{d}x^\nu/\mathrm{d}\lambda}{\sqrt{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}}}\right) - \frac{\partial_\kappa g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}}{2\sqrt{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}}}$$

But

$$\sqrt{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda = \mathrm{d}s,$$

so dividing through by $\sqrt{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}}$ we can replace each $\mathrm{d}\lambda$ with $\mathrm{d}s$ and have

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\left(g_{\kappa\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\right) - \tfrac{1}{2}\partial_\kappa g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}.$$

Multiplying through by $g^{\gamma\kappa}$ and cleaning up

$$0 = \frac{\mathrm{d}^2x^\gamma}{\mathrm{d}s^2} + \tfrac{1}{2}g^{\gamma\kappa}\left(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}$$

2.

$$\begin{aligned} A_{\mu;\nu} &= \partial_\nu A_\mu - \Gamma^\alpha_{\nu\mu}A_\alpha \\ A_{\nu;\mu} &= \partial_\mu A_\mu - \Gamma^\alpha_{\mu\nu}A_\alpha \end{aligned}$$

On subtracting the two equations the terms with Γ cancel because the Christoffel symbol is symmetric in its subscripts.

3.

$$\Gamma^\mu_{\mu\nu} = \tfrac{1}{2}g^{\mu\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})$$

But interchanging the labels μ and α we see that

$$g^{\mu\alpha}\partial_\mu g_{\alpha\nu} = g^{\alpha\mu}\partial_\alpha g_{\mu\nu} = g^{\mu\alpha}\partial_\alpha g_{\mu\nu}$$

so the first and last terms in the brackets cancel and we have $\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\alpha}\partial_\nu g_{\mu\alpha}$.

From the definition of a cofactor,

$$\frac{\partial\det(\mathbf{A})}{\partial A_{ij}} = \mathrm{cof}(A_{ij}) \quad \Rightarrow \quad \delta\ln(\det(\mathbf{A})) = \sum_{ij}\frac{1}{\det(\mathbf{A})}\mathrm{cof}(A_{ij})\delta A_{ij}.$$

Now by Kramer's rule $(A^{-1})_{ij} = \mathrm{cof}(A_{ji})/\det(\mathbf{A})$, so

$$(\mathbf{A}^{-1}\cdot\delta\mathbf{A})_{ik} = \sum_j\frac{\mathrm{cof}(A_{ji})}{\det(\mathbf{A})}\delta A_{jk} \quad \Rightarrow \quad \mathrm{Tr}\,\mathbf{A}^{-1}\cdot\delta\mathbf{A} = \sum_{jk}\frac{\mathrm{cof}(A_{jk})}{\det(\mathbf{A})}\delta A_{jk}$$

which completes the proof that $\delta\ln(\det(\mathbf{A})) = \mathrm{Tr}\,\mathbf{A}^{-1}\cdot\delta\mathbf{A}$.

$$\begin{aligned} \partial_\nu(\ln\sqrt{g}) &= \tfrac{1}{2}\partial_\nu\ln g = \tfrac{1}{2}\,\mathrm{Tr}\left(\mathbf{g}^{-1}\cdot\partial_\nu\mathbf{g}\right) \\ &= \tfrac{1}{2}g^{\alpha\beta}\partial_\nu g_{\alpha\beta} = \Gamma^\mu_{\mu\nu}. \end{aligned}$$

4.

$$
\begin{aligned}
g_{\mu\nu} \to g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = (\delta^\alpha_\mu - \partial_\mu \xi^\alpha)(\delta^\beta_\nu - \partial_\nu \xi^\beta) g_{\alpha\beta} \\
&= g_{\mu\nu} - \partial_\mu \xi^\alpha g_{\alpha\nu} - \partial_\nu \xi^\beta g_{\mu\beta} + \mathrm{O}(\boldsymbol{\xi}^2)
\end{aligned} \tag{1}
$$

Hence

$$
\begin{aligned}
h'_{\mu\nu} &= h_{\mu\nu} - \partial_\mu \xi^\alpha \eta_{\alpha\nu} - \partial_\nu \xi^\beta \eta_{\mu\beta} + \mathrm{O}(\boldsymbol{\xi}^2) \\
&\simeq h_{\mu\nu} - \partial_\mu \xi_\alpha - \partial_\nu \xi_\beta .
\end{aligned}
$$

In e.m., $A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha \Lambda$. The first line of equation (1) gives the formula for the strong-field case.

5. We start from the definition

$$
\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta})
$$

We multiply it by the matrices required to transform each index on the left and use the chain rule on the right:

$$
\frac{\partial x'^\kappa}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}\frac{\partial x'^\kappa}{\partial x^\mu} g^{\mu\nu}\left(\frac{\partial x^\beta}{\partial x'^\delta}\frac{\partial g_{\nu\beta}}{\partial x'^\gamma} + \frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial g_{\alpha\nu}}{\partial x'^\delta} - \frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right)
$$

Into the sum over ν we now insert

$$
\delta^\sigma_\nu = \frac{\partial x'^\lambda}{\partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}
$$

and get

$$
\begin{aligned}
\mathrm{LHS} &= \tfrac{1}{2} g'^{\kappa\lambda}\left(\frac{\partial x^\sigma}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\delta}\frac{\partial g_{\sigma\beta}}{\partial x'^\gamma} + \frac{\partial x^\sigma}{\partial x'^\lambda}\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial g_{\alpha\sigma}}{\partial x'^\delta} - \frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}\frac{\partial g_{\alpha\beta}}{\partial x'^\lambda}\right) \\
&= \Gamma'^\kappa_{\gamma\delta} - \tfrac{1}{2} g'^{\kappa\lambda}\left[g_{\sigma\beta}\frac{\partial}{\partial x'^\gamma}\left(\frac{\partial x^\sigma}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\delta}\right) + g_{\alpha\sigma}\frac{\partial}{\partial x'^\delta}\left(\frac{\partial x^\sigma}{\partial x'^\lambda}\frac{\partial x^\alpha}{\partial x'^\gamma}\right) - g_{\alpha\beta}\frac{\partial}{\partial x'^\lambda}\left(\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}\right)\right]
\end{aligned}
$$

In the first terms we can replace σ by α and in the second we can replace σ by β. Then differentiating out the products oth the terms spawned by the third term cancel on terms spawned by the first two terms, and we have

$$
\frac{\partial x'^\kappa}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}\Gamma^\mu_{\alpha\beta} = \Gamma'^\kappa_{\gamma\delta} - g'^{\kappa\lambda} g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial^2 x^\beta}{\partial x'^\gamma \partial x'^\delta}.
$$

Into the sum over β we now insert

$$
\delta^\beta_\epsilon = \frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\epsilon} \tag{1}
$$

thus

$$
\frac{\partial x'^\kappa}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}\Gamma^\mu_{\alpha\beta} = \Gamma'^\kappa_{\gamma\delta} - g'^{\kappa\lambda} g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\epsilon}\frac{\partial^2 x^\epsilon}{\partial x'^\gamma \partial x'^\delta}.
$$

We now identify $g'_{\lambda\mu}$ and exploit the fact that it is the inverse of $g'^{\kappa\lambda}$ to arrive at

$$
\frac{\partial x'^\kappa}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}\Gamma^\mu_{\alpha\beta} = \Gamma'^\kappa_{\gamma\delta} - \frac{\partial x'^\kappa}{\partial x^\epsilon}\frac{\partial^2 x^\epsilon}{\partial x'^\gamma \partial x'^\delta}. \tag{2}
$$

Differentiating (1) we have

$$
\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\lambda}\frac{\partial x'^\lambda}{\partial x^\eta} = -\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial^2 x'^\lambda}{\partial x'^\mu \partial x^\beta} = -\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\sigma \partial x^\beta}
$$

Using this to replace the double derivative in (2) and rearranging we get the required expression

From the given expression we have

$$\begin{aligned}\Gamma'^\lambda = g'^{\mu\nu}\Gamma'^\lambda_{\mu\nu} &= g'^{\mu\nu}\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} - g'^{\mu\nu}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}\\ &= g^{\tau\sigma}\frac{\partial x'^\lambda}{\partial x^\rho}\Gamma^\rho_{\tau\sigma} - g^{\sigma\rho}\frac{\partial^2 x'^\lambda}{\partial x^\rho\partial x^\sigma}\\ &= \Gamma^\rho\frac{\partial x'^\lambda}{\partial x^\rho} - g^{\sigma\rho}\frac{\partial^2 x'^\lambda}{\partial x^\rho\partial x^\sigma}\end{aligned}$$

With the left side set to zero this becomes a set of linear p.d.es for x'^λ that we should be able to solve for given source terms $\Gamma^\rho(\mathbf{x})$ in the same way that we would solve the wave equation for the vector potential A^λ generated by a given current density j^ρ.

We always have

$$0 = \nabla_\kappa g^{\lambda\kappa} = \partial_\kappa g^{\lambda\kappa} + \Gamma^\lambda_{\kappa\alpha}g^{\alpha\kappa} + \Gamma^\kappa_{\kappa\alpha}g^{\lambda\alpha}$$

so

$$\partial_\kappa g^{\lambda\kappa} = -\Gamma^\lambda - g^{\lambda\alpha}\partial_\alpha \ln(\sqrt{g})$$

where we've used eq. (†) of the problem set. Now

$$\partial_\kappa\left(\sqrt{g}g^{\lambda\kappa}\right) = \partial_\kappa\left(\mathrm{e}^{\ln\sqrt{g}}g^{\lambda\kappa}\right) = \sqrt{g}\left[\partial_\kappa(\ln\sqrt{g})g^{\lambda\kappa} - \Gamma^\lambda - g^{\lambda\alpha}\partial_\alpha\ln\sqrt{g}\right] = -\sqrt{g}\,\Gamma^\lambda$$

so $\partial_\kappa\left(\sqrt{g}g^{\lambda\kappa}\right) = 0$ in the harmonic gauge.

$$\begin{aligned}\Box\phi = \nabla_\kappa g^{\kappa\lambda}\partial_\lambda\phi &= \partial_\kappa(g^{\kappa\lambda}\partial_\lambda\phi) + \Gamma^\kappa_{\kappa\beta}g^{\beta\lambda}\partial_\lambda\phi\\ &= g^{\kappa\lambda}\frac{\partial^2\phi}{\partial x^\kappa\partial x^\lambda} + \partial_\lambda\phi\left(\partial_\kappa g^{\kappa\lambda} + g^{\beta\lambda}\partial_\beta\ln\sqrt{g}\right)\end{aligned}$$

But

$$\partial_\kappa g^{\kappa\lambda} + g^{\beta\lambda}\partial_\beta\ln\sqrt{g} = \partial_\kappa g^{\kappa\lambda} + \frac{g^{\kappa\lambda}\partial_\kappa\sqrt{g}}{\sqrt{g}} = \frac{1}{\sqrt{g}}\partial_\kappa\left(\sqrt{g}g^{\kappa\lambda}\right) = 0.$$

6.

$$\begin{aligned}g_{\beta\lambda}\partial_\nu\Gamma^\lambda_{\alpha\gamma} &= g_{\beta\lambda}\partial_\nu\left(g^{\lambda\sigma}\Gamma_{\sigma\alpha\gamma}\right)\\ &= g_{\beta\lambda}\partial_\nu(g^{\lambda\sigma})\Gamma_{\sigma\alpha\gamma} + \delta^\sigma_\beta\partial_\nu\Gamma_{\sigma\alpha\gamma}\\ &= -g^{\lambda\sigma}\left(\Gamma^\eta_{\nu\beta}g_{\eta\lambda} + \Gamma^\eta_{\nu\lambda}g_{\eta\beta}\right)\Gamma_{\sigma\alpha\gamma} + \partial_\nu\Gamma_{\beta\alpha\gamma}\\ &= -\Gamma^\lambda_{\alpha\gamma}\left(\Gamma^\eta_{\nu\beta}g_{\eta\lambda} + \Gamma^\eta_{\nu\lambda}g_{\eta\beta}\right) + \partial_\nu\Gamma_{\beta\alpha\gamma}\end{aligned}$$

We form $R_{\beta\gamma\alpha\nu}$ by antisymmetrizing this in $\alpha\nu$ and adding the $\Gamma\Gamma$ terms, so

$$R_{\beta\gamma\alpha\nu} = \partial_\nu\Gamma_{\beta\alpha\gamma} - \partial_\alpha\Gamma_{\beta\nu\gamma} - \Gamma^\lambda_{\alpha\gamma}\left(\Gamma^\eta_{\nu\beta}g_{\eta\lambda} + \Gamma^\eta_{\nu\lambda}g_{\eta\beta}\right) + \Gamma^\lambda_{\nu\gamma}\left(\Gamma^\eta_{\alpha\beta}g_{\eta\lambda} + \Gamma^\eta_{\alpha\lambda}g_{\eta\beta}\right) + g_{\beta\mu}\left(\Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\alpha\gamma} - \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\nu\gamma}\right)$$

On cancelling the second $\Gamma\Gamma$ term on the fifth and the fourth on the sixth we obtain the desired $\Gamma\Gamma$ terms. Similarly when we replace the Γs with all subscripts by their definitions in terms of derivatives of $\mathbf{g}$ two of the six double derivatives cancel and we are left with the desired terms.

The permutation $\mu \to \nu \to \kappa \to \mu$ interchanges the first and the third double derivatives in the given expression so two pairs of these terms will cancel between permutations. The other two double derivatives are ∂_λ of $\partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}$, which changes sign under $\kappa \to \nu \to \mu$, so again terms in the cyclic sum cancel. The $\Gamma\Gamma$ terms have the same symmetry properties and likewise cancel. This completes the proof of the cyclic identity $R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0$.

$R_{\lambda\mu\nu\kappa}$ can be thought of as R_{AB} where A and B are antisymmetric pairs of indices. There are $\frac{1}{2}4 \times 3 = 6$ such possible pairs. Also $R_{AB} = R_{BA}$ so there are $\frac{1}{2}6 \times 7 = 21$ independent ways of choosing a distinct pair of index-pairs. Using these pair-wise symmetries we can show that the cyclic sum $S_{\alpha\beta\delta\gamma} \equiv R_{\alpha\beta\delta\gamma} + R_{\alpha\delta\gamma\beta} + R_{\alpha\gamma\beta\delta}$ is completely antisymmetric. For example,

$$\begin{aligned} S_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} \\ &= -R_{\alpha\beta\delta\gamma} - R_{\alpha\gamma\beta\delta} - R_{\alpha\delta\gamma\beta} = -S_{\alpha\beta\gamma\delta} \end{aligned}$$

Hence already from the pair-wise symmetries of $\mathbf{R}$ we can show that it is non-zero only when all four indices are different. Thus the cyclic identity adds the single constraint $S_{01234} = 0$ on the components of $\mathbf{R}$, and there are 20 independent components overall.

7. From the previous question

$$\begin{aligned} R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa} &\simeq \tfrac{1}{2}\eta^{\lambda\nu} \left(\frac{\partial^2 h_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right) \\ &= \tfrac{1}{2}[\partial_\mu \partial_\kappa h - \partial^\nu(\partial_\kappa h_{\mu\nu} + \partial_\mu h_{\nu\kappa} - \partial_\nu h_{\mu\kappa})] \end{aligned} \tag{3}$$

as required.

The gauge condition is

$$\begin{aligned} 0 = 2g^{\mu\nu}\Gamma^\lambda_{\mu\nu} &\simeq \eta^{\mu\nu}\eta^{\lambda\kappa}(\partial_\mu h_{\kappa\nu} + \partial_\nu h_{\mu\kappa} - \partial_\kappa h_{\mu\nu}) \\ &= 2\partial_\mu h^{\mu\lambda} - \partial^\lambda h. \end{aligned}$$

Plugging

$$\partial^\kappa h_{\lambda\kappa} = \tfrac{1}{2}\partial_\lambda h$$

into (3) we get

$$R_{\mu\kappa} \simeq \tfrac{1}{2}[\partial_\mu\partial_\kappa h - \tfrac{1}{2}(\partial_\mu\partial_\kappa h + \partial_\mu\partial_\kappa h) + \Box h_{\mu\kappa}] = \tfrac{1}{2}\Box h_{\mu\kappa}$$

Taking the trace of this equation we find that $R = \frac{1}{2}\Box h$, so the Einstein equations are

$$R_{\mu\kappa} - \tfrac{1}{2}R\eta_{\mu\kappa} = \tfrac{1}{2}\Box\overline{h}_{\mu\kappa} = -\frac{8\pi G}{c^4}T_{\mu\kappa} \tag{4}$$

The analogous e.m. equation is $\Box A^\mu = \mu_0 j^\mu$. From this equation we easily infer the propagation of e.m. waves, so in (4) we have an equation that predicts gravitational waves.

8. Now for stationary rest mass the only non-vanishing element of $T_{\mu\kappa}$ is $T_{00} = \rho c^2$. So $\overline{h}_{\mu\kappa} = 0$ for $\mu\kappa \neq 00$. It follows that $h_{\mu\kappa} = \frac{1}{2}h\eta_{\mu\kappa}$ for $\mu\kappa \neq 00$. Now $h = -h_{00} + h_{ii} = -h_{00} + \frac{3}{2}h$, so $h = 2h_{00}$. Moreover $\overline{h}_{00} = h_{00} + \frac{1}{2}h = 2h_{00}$. Thus h_{00} satisfies the equation

$$\Box h_{00} = -\frac{8\pi G}{c^2}\rho$$

Bearing in mind that the matter is stationary so $\partial_0 \mathbf{h} = 0$, and Poisson's equation $\nabla^2\Phi = 4\pi G\rho$, we see that $h_{00} = -2\Phi/c^2$. Finally the other non-zero elements of $\mathbf{h}$ are $h_{xx} = h_{yy} = h_{zz} = \frac{1}{2}h = h_{00} = -2\Phi/c^2$ as required.