Classical Fields I: Solutions

1. In the centre of mass of the pre-decay particle and with a suitable orientation of axes, the photons have wavevectors $k_\pm^\mu = k(1, \pm\cos\theta, \pm\sin\theta, 0)$. In the lab frame the wavevectors are

$$K_\pm^\mu = k\begin{pmatrix}\gamma & \beta\gamma & 0 & \\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ \pm\cos\theta\\ \pm\sin\theta\\ 1\end{pmatrix} = k\begin{pmatrix}\gamma\pm\beta\gamma\cos\theta\\ \beta\gamma\pm\gamma\cos\theta\\ \pm\sin\theta\\ 0\end{pmatrix}$$

We get the cosine of the angle between the lines of flight of the photons from the dot product of the spatial wavevectors. Exploiting the fact that $\mathbf{K}$ is still a null vector to fix the normalization, we have

$$\begin{aligned}\cos\alpha &= \frac{(\beta\gamma-\gamma\cos\theta, -\sin\theta, 0)}{\gamma(1-\beta\cos\theta)}\cdot\frac{(\beta\gamma+\gamma\cos\theta, \sin\theta, 0)}{\gamma(1+\beta\cos\theta)}\\ &= \frac{\gamma^2(\beta^2-\cos^2\theta)-\sin^2\theta}{\gamma^2(1-\beta^2\cos^2\theta)}\\ &= \frac{\beta^2(1+\sin^2\theta)-1}{1-\beta^2\cos^2\theta}.\end{aligned}$$

Solving for $\sin^2\theta$ we find

$$\sin^2\theta = \frac{(1-\beta^2)(1+\cos\alpha)}{\beta^2(1-\cos\alpha)} = \frac{1-\beta^2}{\beta^2}\cot^2\alpha/2 = \frac{1}{\gamma^2\beta^2}\cot^2\alpha/2.$$

Since the meson was spin zero, the decay must be isotropic in its rest frame and $\mathrm{d}N = \sin\theta\,\mathrm{d}\theta$. Differentiating $\sin\theta = (\gamma\beta)^{-1}\cot\alpha/2$ wrt θ we have

$$\begin{aligned}\mathrm{d}N &= \frac{1}{\gamma\beta}\cot\tfrac12\alpha\,\frac{1}{\gamma\beta}\csc^2\tfrac12\alpha\frac{\mathrm{d}\alpha}{2\cos\theta}\\ &= \frac{\mathrm{d}\alpha}{2\gamma^2\beta^2}\frac{\cos\frac12\alpha}{\sin^3\frac12\alpha\sqrt{1-(\gamma\beta)^{-2}\cot^2\frac12\alpha}}\\ &= \frac{\mathrm{d}\alpha\,\sin\alpha}{4\gamma^2\beta\sin^3\frac12\alpha(\beta^2-\cos^2\frac12\alpha)^{1/2}}\end{aligned}$$

2. In the frame of the conductor there is no E field inside the conductor and the B field is the same as outside it. So we first find $\mathbf{B}$ in the conductor's rest frame by transforming $(\mathbf{B}, \mathbf{E}) = (B, 0)$ to the boosted frame

$$\boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel \qquad \boldsymbol{B}'_\perp = \gamma\boldsymbol{B}_\perp$$

We now set $\boldsymbol{E}' = 0$ and transform back to the lab frame

$$\begin{aligned}&\boldsymbol{B}''_\parallel = \boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel \qquad &&\boldsymbol{B}''_\perp = \gamma\boldsymbol{B}'_\perp = \gamma^2\boldsymbol{B}_\perp\\ &\boldsymbol{E}''_\parallel = 0 &&\boldsymbol{E}''_\perp = -\gamma(\boldsymbol{v}\times\boldsymbol{B}'_\perp) = -\gamma^2(\boldsymbol{v}\times\boldsymbol{B}_\perp)\end{aligned}$$

3. We need to make a Lorentz scalar from $\mathbf{p}_a$, $\mathbf{p}_b$ and $\mathbf{F}$. Only 2 possibilities arise: $\mathbf{p}_a\cdot\mathbf{F}\cdot\mathbf{p}_b$ and $\mathbf{p}_a\cdot\overline{\mathbf{F}}\cdot\mathbf{p}_b$, but only the first is a true rather than a pseudo scalar. With $\boldsymbol{E} = 0$ we have $\boldsymbol{p}_a\times\boldsymbol{p}_b\cdot\boldsymbol{B}$, which is a true scalar. So $X \propto \mathbf{p}_a\cdot\mathbf{F}\cdot\mathbf{p}_b$. To determine the constant of proportionality we examine the case $\boldsymbol{E} = 0$:

$$\begin{aligned}\mathbf{p}_a\cdot\mathbf{F}\cdot\mathbf{p}_b &= \mathbf{p}_a\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & B_z & -B_y\\ 0 & -B_z & 0 & B_x\\ 0 & B_y & -B_x & 0\end{pmatrix}\begin{pmatrix}p_b^0\\ p_b^x\\ p_b^y\\ p_b^z\end{pmatrix}\\ &= (p_a^0, p_a^x, p_a^y, p_a^z)(0, B_zp_b^y - B_yp_b^z, -B_zp_b^x + B_xp_b^z, B_yp_b^x - B_xp_b^y)\\ &= \boldsymbol{p}_a\cdot(\boldsymbol{p}_b\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{p}_a\times\boldsymbol{p}_b)\end{aligned}$$

so the constant of proportionality is unity.

4. From the Lorentz force

$$\boldsymbol{f} = \int \mathrm{d}^3\boldsymbol{r}(\rho\boldsymbol{E} + \boldsymbol{j}\times\boldsymbol{B}).$$

With Maxwell's eqns

$$\nabla\cdot\boldsymbol{D} = \rho \qquad \nabla\times\boldsymbol{H} = \boldsymbol{j} + \frac{\partial\boldsymbol{D}}{\partial t}$$
$$\nabla\cdot\boldsymbol{B} = 0 \qquad \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$

this becomes

$$\boldsymbol{f} = \int \mathrm{d}^3\boldsymbol{r}\left[(\nabla\cdot\boldsymbol{D})\boldsymbol{E} + \left(\nabla\times\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t}\right)\times\boldsymbol{B}\right].$$

Now

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}(\boldsymbol{D}\times\boldsymbol{B}) &= \int \mathrm{d}^3\boldsymbol{r}\left(\frac{\partial\boldsymbol{D}}{\partial t}\times\boldsymbol{B} + \boldsymbol{D}\times\frac{\partial\boldsymbol{B}}{\partial t}\right)\\ &= \int \mathrm{d}^3\boldsymbol{r}\left(\frac{\partial\boldsymbol{D}}{\partial t}\times\boldsymbol{B} - \boldsymbol{D}\times(\nabla\times\boldsymbol{E})\right)\end{aligned}$$

so

$$\boldsymbol{f} = \int \mathrm{d}^3\boldsymbol{r}\left[(\nabla\cdot\boldsymbol{D})\boldsymbol{E} + (\nabla\times\boldsymbol{H})\times\boldsymbol{B} - \boldsymbol{D}\times(\nabla\times\boldsymbol{E})\right] - \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}(\boldsymbol{D}\times\boldsymbol{B})$$

as required. Now

$$\begin{aligned}\boldsymbol{n}\cdot\boldsymbol{W} &= (\boldsymbol{n}\cdot\boldsymbol{E})(\nabla\cdot\boldsymbol{D}) - \boldsymbol{n}\cdot[\boldsymbol{B}\times(\nabla\times\boldsymbol{H})] - \boldsymbol{n}\cdot[\boldsymbol{D}\times(\nabla\times\boldsymbol{E})]\\ &= (\boldsymbol{n}\cdot\boldsymbol{E})(\nabla\cdot\boldsymbol{D}) - [B_j(\boldsymbol{n}\cdot\nabla)H_j - (\boldsymbol{B}\cdot\nabla)(\boldsymbol{n}\cdot\boldsymbol{H}) + D_j(\boldsymbol{n}\cdot\nabla)E_j - \boldsymbol{D}\cdot\nabla(\boldsymbol{n}\cdot\boldsymbol{E})]\end{aligned}$$

When $\mu_0\boldsymbol{H} = \boldsymbol{B}$ and $\boldsymbol{D} = \epsilon_0\boldsymbol{E}$ this becomes

$$\boldsymbol{n}\cdot\boldsymbol{W} = \epsilon_0(\boldsymbol{n}\cdot\boldsymbol{E})\nabla\boldsymbol{E} - \tfrac{1}{2}\boldsymbol{n}\cdot\nabla B^2/\mu_0 + \boldsymbol{B}\cdot\nabla(\boldsymbol{n}\cdot\boldsymbol{B}/\mu_0) - \tfrac{1}{2}\epsilon_0\boldsymbol{n}\cdot\nabla E^2 + \epsilon_0\boldsymbol{E}\cdot\nabla(\boldsymbol{n}\cdot\boldsymbol{E})$$

which agrees with $\boldsymbol{n}\cdot\boldsymbol{W}$. But

$$\nabla\cdot\boldsymbol{U}(\boldsymbol{n}) = \epsilon_0\left[\boldsymbol{E}\cdot\nabla(\boldsymbol{n}\cdot\boldsymbol{E}) + (\boldsymbol{n}\cdot\boldsymbol{E})\nabla\cdot\boldsymbol{E} - \tfrac{1}{2}\boldsymbol{n}\cdot\nabla E^2\right] + \frac{1}{\mu_0}\left[\boldsymbol{B}\cdot\nabla(\boldsymbol{n}\cdot\boldsymbol{B}) + (\boldsymbol{n}\cdot\boldsymbol{B})\nabla\cdot\boldsymbol{B} - \tfrac{1}{2}\boldsymbol{n}\cdot\nabla B^2\right]$$

which agrees with $\boldsymbol{n}\cdot\boldsymbol{W}$. $\boldsymbol{D}\times\boldsymbol{B}$ is the momentum density of the e.m. field. $\boldsymbol{U}(\boldsymbol{n})$ is the flux of momentum across a surface with normal $\boldsymbol{n}$. Hence $\boldsymbol{U} = \mathbf{T}\cdot\mathbf{n}$, where $\mathbf{n} = (0, n_x, n_y, n_z)$.

5. Consider

$$X_{ij} = \eta_i\eta_j^* = \begin{pmatrix}\eta_1\eta_1^* & \eta_1\eta_2^*\\ \eta_1^*\eta_2 & \eta_2\eta_2^*\end{pmatrix}$$

$\det(X) = |\eta_1|^2|\eta_2|^2 - \eta_1^*\eta_2\eta_1\eta_2^* = 0$ so we must have $|\mathbf{v}|^2 = 0$. Thus only null vectors can be represented by X.

6. This problem can be done by explicitly multiplying the 2×2 matrices. Here's a more cerebral solution

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta\sigma_{\boldsymbol{n}}}x^\mu\sigma_\mu\mathrm{e}^{-\mathrm{i}\theta\sigma_{\boldsymbol{n}}} &= x^\mu(\cos\theta + \mathrm{i}\sin\theta\sigma_{\boldsymbol{n}})\sigma_\mu(\cos\theta - \mathrm{i}\sin\theta\sigma_{\boldsymbol{n}})\\ &= x^\mu\left[\cos^2\theta\sigma_\mu + \sin^2\theta\sigma_{\boldsymbol{n}}\sigma_\mu\sigma_{\boldsymbol{n}} + \mathrm{i}\sin\theta\cos\theta(\sigma_{\boldsymbol{n}}\sigma_\mu - \sigma_\mu\sigma_{\boldsymbol{n}})\right]\end{aligned}$$

Now

$$\sigma_{\boldsymbol{n}}\sigma_\mu - \sigma_\mu\sigma_{\boldsymbol{n}} = n_i[\sigma_i, \sigma_\mu] = \begin{cases}0 & \text{if } \mu = 0\\ 2\mathrm{i}n_i\epsilon_{i\mu k}\sigma_k & \text{if } \mu = 1,2,3\end{cases}$$

Hence

$$\begin{aligned}\sigma_{\boldsymbol{n}}\sigma_\mu\sigma_{\boldsymbol{n}} &= \sigma_{\boldsymbol{n}}^2\sigma_\mu - \sigma_{\boldsymbol{n}}[\sigma_\mu, \sigma_{\boldsymbol{n}}]\\ &= \sigma_\mu - \begin{cases}0 & \text{if } \mu = 0\\ 2\mathrm{i}n_i\epsilon_{i\mu k}\sigma_{\boldsymbol{n}}\sigma_k & \text{if } \mu = 1,2,3\end{cases}\\ &= \sigma_\mu - \begin{cases}0 & \text{if } \mu = 0\\ 2\mathrm{i}n_i n_j\epsilon_{i\mu k}\sigma_j\sigma_k & \text{if } \mu = 1,2,3\end{cases}\end{aligned}$$

Also $\sigma_j\sigma_k = \mathrm{i}\epsilon_{ljk}\sigma_l$, so

$$\begin{aligned}\sigma_{\boldsymbol{n}}\sigma_\mu\sigma_{\boldsymbol{n}} &= \sigma_\mu - \begin{cases} 0 & \text{if } \mu=0 \\ -2\epsilon_{i\mu k}\epsilon_{ljk}n_i n_j\sigma_l & \text{if } \mu=1,2,3\end{cases}\\ &= \sigma_\mu - \begin{cases} 0 & \text{if } \mu=0 \\ -2\sigma_{\boldsymbol{n}}n_\mu + 2\sigma_\mu & \text{if } \mu=1,2,3\end{cases}\\ &= \begin{cases} \sigma_\mu & \text{if } \mu=0 \\ -\sigma_\mu + 2\sigma_{\boldsymbol{n}}n_\mu & \text{if } \mu=1,2,3\end{cases}\end{aligned}$$

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta\sigma_{\boldsymbol{n}}}x^\mu\sigma_\mu\mathrm{e}^{-\mathrm{i}\theta\sigma_{\boldsymbol{n}}} &= x^0\sigma_0 + x^l\left[(\cos^2\theta-\sin^2\theta)\sigma_l + 2\sigma_{\boldsymbol{n}}n_l\sin^2\theta + \mathrm{i}\tfrac{1}{2}\sin 2\theta\, 2\mathrm{i}n_i\epsilon_{ilk}\sigma_k\right]\\ &= x^0\sigma_0 + \cos 2\theta\,\boldsymbol{x}\cdot\sigma + (1-\cos 2\theta)(\boldsymbol{n}\cdot\sigma)\boldsymbol{n}\cdot\boldsymbol{x} - \sin 2\theta(\boldsymbol{n}\times\boldsymbol{x})\cdot\sigma \qquad (\dagger)\\ &= x^0\sigma_0 - \sin 2\theta(\boldsymbol{n}\times\boldsymbol{x})\cdot\sigma + \cos 2\theta\,[\boldsymbol{x}\cdot\sigma - (\boldsymbol{n}\cdot\sigma)(\boldsymbol{n}\cdot\boldsymbol{x})] + (\boldsymbol{n}\cdot\sigma)(\boldsymbol{x}\cdot\boldsymbol{n})\end{aligned}$$

Now any spatial vector $\boldsymbol{x}$ can be resolved into components parallel and perpendicular to a given vector $\boldsymbol{n}$

$$\boldsymbol{x} = (\boldsymbol{x}\cdot\boldsymbol{n})\boldsymbol{n} - \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{x})$$

and when we rotate $\boldsymbol{x}$ about $\boldsymbol{n}$ through angle 2θ the rotated vector is

$$\boldsymbol{x}' = (\boldsymbol{x}\cdot\boldsymbol{n})\boldsymbol{n} - \cos 2\theta\,\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{x}) - \sin 2\theta(\boldsymbol{n}\times\boldsymbol{x})$$

Expanding the vector triple product and then dotting the whole equation with σ, the right side becomes identical with the right side of (†), while the time component of $\mathbf{x}$ is clearly unchanged by the rotation.

7. The E-L eqn is

$$\partial_\nu\frac{\partial\mathcal{L}}{\partial\phi_\nu} - \frac{\partial\mathcal{L}}{\partial\phi} = 0,$$

So $-\partial_\nu\partial^\nu\phi + \mathrm{d}V/\mathrm{d}\phi = 0$. For K-G generalize ϕ to a complex field and take $V = m_0|\phi|^2$.

For $V = 1-\cos\phi$, $-\Box\phi + \sin\phi = 0$. If $\phi = \Phi(x-\beta ct)$, then $\Box\phi = -\Phi''\beta^2 + \Phi'' = (1-\beta^2)\Phi''$. Thus $-(1-\beta^2)\Phi'' + \sin\Phi = 0$. We multiply by Φ' and integrate w.r.t. $X \equiv x - ct$ and have

$$\begin{aligned}&(1-\beta^2)\frac{\mathrm{d}}{\mathrm{d}X}\tfrac{1}{2}(\Phi')^2 + \frac{\mathrm{d}}{\mathrm{d}X}(\cos\Phi) = 0\\ \Rightarrow\quad &(1-\beta^2)\tfrac{1}{2}(\Phi')^2 + \cos\Phi = \text{const}\\ \Rightarrow\quad &(1-\beta^2)\tfrac{1}{2}(\Phi')^2 - 2\sin^2\Phi/2 = \text{const}\end{aligned}$$

For vanishing constant on the right side

$$\begin{aligned}&\Phi' = \pm\frac{2\sin\Phi/2}{\sqrt{1-\beta^2}} \quad\Rightarrow\quad \frac{\mathrm{d}\Phi}{\sin\Phi/2} = \pm\frac{2\mathrm{d}X}{\sqrt{1-\beta^2}}\\ \Rightarrow\quad &\pm 4\gamma\mathrm{d}X = \frac{\mathrm{d}\Phi}{\sin\frac{1}{4}\Phi\cos\frac{1}{4}\Phi} = \frac{\sin\frac{1}{4}\Phi\,\mathrm{d}\Phi}{(1-\cos^2\frac{1}{4}\Phi)\cos\frac{1}{4}\Phi} = -4\mathrm{d}\mu\left(\frac{1}{\mu} + \frac{\mu}{1-\mu^2}\right)\\ \Rightarrow\quad &\pm\gamma(X-X_0) = -\ln\left(\frac{\mu}{\pm\sqrt{1-\mu^2}}\right) + \text{const.} = -\ln\left(\frac{\cos\Phi/4}{\pm\sin\Phi/4}\right) + \text{const.}\end{aligned}$$

Thus

$$A\mathrm{e}^{\pm\gamma(X-X_0)} = \pm\tan(\Phi/4).$$

As $X\to-\infty$, $\Phi\to 0$, and as $X\to\infty$, $\Phi\to 2\pi$. Thus there is a step change in Φ that moves at speed βc.

8.

$$\hat{T}^\mu_\nu = -\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial_\nu \phi - \mathcal{L}\delta^\mu_\nu\right)$$
$$= \partial^\mu \phi \partial_\nu \phi - (\tfrac{1}{2}\partial_\alpha \phi \partial^\alpha \phi + V)\delta^\mu_\nu$$

Multiplying by $\eta^{\nu 0}$ we find

$$\hat{T}^{00} = -\hat{T}^0_0 = -\partial^0 \phi \partial_0 \phi + \tfrac{1}{2}(\partial_0 \phi \partial^0 \phi + \partial_i \phi \partial^i \phi + V)$$
$$= -\tfrac{1}{2}\partial^0 \phi \partial_0 \phi + \tfrac{1}{2}|\nabla \phi|^2 + V = \tfrac{1}{2}[(\partial_0 \phi)^2 + |\nabla \phi|^2] + V.$$

9. $\psi \to e^{i\theta}\psi \Rightarrow \overline{\psi} = \psi^* \gamma^0 \to e^{-i\theta}\overline{\psi}$, so $\mathcal{L} \to e^{-i\theta}\overline{\psi}\gamma^\mu \partial_\mu (e^{i\theta}\psi) - m\overline{\psi}\psi$. If θ is not a function of $\mathbf{x}$, new $\mathcal{L}$ is the old $\mathcal{L}$. The current is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \overline{\psi}}\delta\overline{\psi} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi}\delta\psi.$$

But

$$\partial \mathcal{L}/\partial(\partial_\mu \overline{\psi}) = 0 \quad ; \quad \partial \mathcal{L}/\partial(\partial_\mu \psi) = \overline{\psi}\gamma^\mu \quad \text{and} \quad \delta\psi = i\theta\psi,$$

so $j^\mu = i\theta\overline{\psi}\gamma^\mu\psi$.

10. In vacuo Maxwell's equations are

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{E} = 0, \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

Adding ic times the B eqns to the E eqns we obtain $0 = \nabla \cdot (\boldsymbol{E} + ic\boldsymbol{B}) = \nabla \cdot \psi$ and

$$\nabla \times (\boldsymbol{E} + ic\boldsymbol{B}) = \frac{\partial}{\partial t}\left(\frac{i\boldsymbol{E}}{c} - \boldsymbol{B}\right) = \frac{i}{c}\frac{\partial}{\partial t}(\boldsymbol{E} + ic\boldsymbol{B}) \quad \Rightarrow \quad \frac{\partial \psi}{\partial t} = -ic\nabla \times \psi.$$

The energy density is $\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}B^2/\mu_0 = \frac{1}{2}\epsilon_0|\psi|^2$. The Poynting vector is

$$\boldsymbol{N} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} = \frac{(\boldsymbol{E} - ic\boldsymbol{B}) \times (\boldsymbol{E} + ic\boldsymbol{B})}{2ic\mu_0}$$

as required. If

$$\psi = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{\pm i(kz - \omega t)}$$

then using the reality of $\boldsymbol{E}$ and $\boldsymbol{B}$

$$E_x + icB_x = e^{\pm i(kz-\omega t)} \quad \Rightarrow \quad \begin{cases} E_x = \cos(kz - \omega t) \\ B_x = \pm\frac{1}{c}\sin(kz - \omega t) \end{cases}$$

$$E_y + icB_y = ie^{\pm i(kz-\omega t)} \quad \Rightarrow \quad \begin{cases} E_y = \mp\sin(kz - \omega t) \\ B_y = \frac{1}{c}\cos(kz - \omega t) \end{cases}$$

so the field has $\boldsymbol{E}$ rotating in the xy plane and is a circularly polarized wave.

The Dirac equation is $i\gamma^\mu \partial_\mu \psi - m\psi = 0$. Our equation contains only derivatives so it's analogous to the Dirac equation only for $m = 0$. Also $\gamma^\mu \partial_\mu$ is a different linear combination of derivatives than

$$\frac{\partial}{\partial t} + ic\nabla\times$$

but is otherwise similar.