Solutions Exercises in Chapter 1

1. *Obtain (1.12) from the requirement that for any two vectors* $\mathbf{x}$, $\mathbf{y}$, *we have* $x'_\mu y'^\mu = x_\mu y^\mu$.
We require

$$x'_\mu y'^\mu = \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta x_\alpha y^\beta = x_\mu y^\mu \qquad \forall\ x, y$$

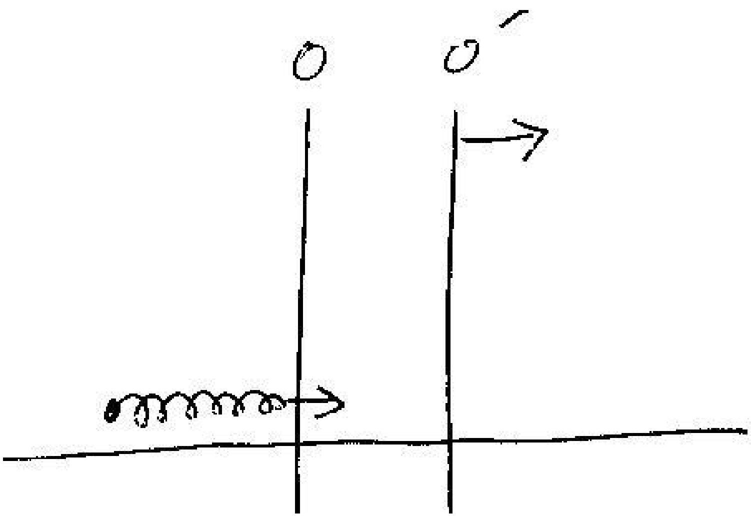
so $\left(\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta - \delta^\alpha_\beta\right) x_\alpha y^\beta = 0 \quad \forall\ x, y.$

By taking $x^\mu = (1, 0, 0, 0)$ etc we can make $x_\alpha y^\beta$ have only one non-zero component at a time. So every component of the bracket has to vanish.

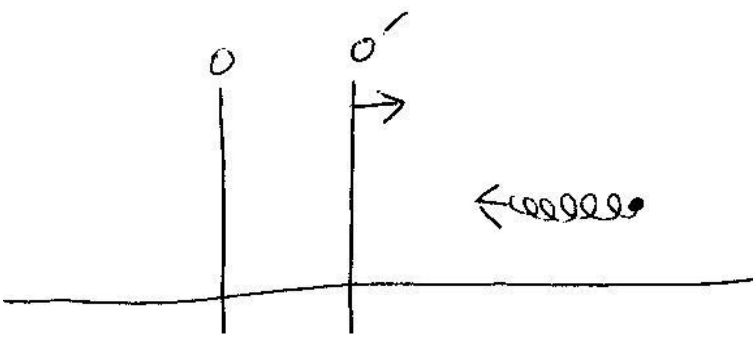
2. *Determine whether the photon is blue or red shifted between its emission by O and its detection by O′. Relate this to the question of whether O′ is approaching or receding from O.*
$\omega'/c = \gamma\omega/c - \beta\gamma k_x$. Suppose $k_x = \omega/c$ (photon moving down +ve x direction). Then

$$\omega'/\omega = \gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}} < 1 \quad \text{so we have a redshift.}$$

Now $x'_1 = \gamma(x_1 - \beta ct)$ so O′ is moving in the +ve x direction. The situation must be like this



so we expect a redshift. Conversely, if $k_x = -\omega/c$, we find $\omega'/\omega = \gamma(1+\beta) > 1$ and the photon is blueshifted. Physically, we have



so O′ is running against the oncoming photon.

3. *Transform $F^{\kappa\lambda}$ with the matrix $\Lambda^\mu{}_\nu$ to show that an observer who moves at speed v down the x-axis of an observer who sees fields $\boldsymbol{E} = (E_x, E_y, 0)$ and $\boldsymbol{B} = 0$, perceives fields $\boldsymbol{E}' = (E_x, \gamma E_y, 0)$ and $\boldsymbol{B}' = (0, 0, \gamma v E_y/c)$. [Hint: since $\mathbf{\Lambda}$ is symmetric, we can write $\mathbf{F}' = \mathbf{\Lambda}\cdot\mathbf{F}\cdot\mathbf{\Lambda}$.] Hence deduce the general rules $\boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel$, $\boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v}\times\boldsymbol{B})$, $\boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel$, $\boldsymbol{B}_\perp = \gamma(\boldsymbol{B}_\perp - \boldsymbol{v}\times\boldsymbol{E}/c^2)$. Verify that $(B^2 - E^2/c^2) = (B'^2 - E'^2/c^2)$.*

$$
\begin{aligned}
F'^{\mu\nu} &= \Lambda \begin{pmatrix} 0 & E_x/c & E_y/c & 0 \\ -E_x/c & 0 & \dots & \\ -E_y/c & 0 & \dots & \\ 0 & & & \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\beta\gamma E_x/c & \gamma E_x/c & E_y/c & 0 \\ -\gamma E_x/c & \beta\gamma E_x/c & 0 & 0 \\ -\gamma E_y/c & \beta\gamma E_y/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \gamma^2\left(\dfrac{E_x}{c} - \beta^2\dfrac{E_x}{c}\right) & \gamma E_y/c & 0 \\ -\gamma^2\dfrac{E_x}{c}(1-\beta^2) & 0 & -\beta\gamma\dfrac{E_y}{c} & 0 \\ -\gamma\dfrac{E_y}{c} & \beta\gamma\dfrac{E_y}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & E_x/c & \gamma E_y/c & 0 \\ -E_x/c & 0 & -\beta\gamma E_y/c & 0 \\ -\gamma E_y/c & \beta\gamma E_y/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Since we can orient the coordinates such that the y axis coincides with the component of $\mathbf{E}$ that is $\perp$ to $\mathbf{v}$, we can conclude that when $\mathbf{B} = 0$

$$
\begin{aligned}
\mathbf{E}' &= \mathbf{E}_\parallel + \gamma\mathbf{E}_\perp \\
\mathbf{B}' &= -(\gamma/c^2)\mathbf{v}\times\mathbf{E}.
\end{aligned}
$$

Similarly, by transforming a pure $\mathbf{B}$ field $(B_x, B_y, 0)$, we discover that

$$
\begin{aligned}
\mathbf{E}' &= \gamma\mathbf{v}\times\mathbf{B} \\
\mathbf{B}' &= \mathbf{B}_\parallel + \gamma\mathbf{B}_\perp.
\end{aligned}
$$

We now argue that for general $(\mathbf{B}, \mathbf{E})$ we can break $\mathbf{F}$ into two parts, one with $\mathbf{E}$ only and one with $\mathbf{B}$ only, transform each as above and recombine. This procedure yields the stated rules.

Consider

$$
\begin{aligned}
B'^2 - E'^2/c^2 &= B'^2_\perp + B'^2_\parallel - (E'^2_\perp + E'^2_\parallel)/c^2 \\
&= \gamma^2\left(\mathbf{B}_\perp - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right)^2 + B^2_\parallel - \frac{1}{c^2}\left[\gamma^2(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B})^2 + E^2_\parallel\right] \\
&= \gamma^2\left[B^2_\perp - 2\mathbf{B}_\perp\cdot\frac{\mathbf{v}\times\mathbf{E}}{c^2} + \frac{|\mathbf{v}\times\mathbf{E}|^2}{c^4} - \frac{E^2_\perp}{c^2} - 2\frac{\mathbf{E}_\perp}{c^2}\cdot\mathbf{v}\times\mathbf{B} - \frac{|\mathbf{v}\times\mathbf{B}|^2}{c^2}\right] \\
&\quad + B^2_\parallel - \frac{E^2_\parallel}{c^2}
\end{aligned}
$$

Now $\mathbf{B}_\perp\cdot\mathbf{v}\times\mathbf{E} = \mathbf{E}\cdot\mathbf{B}_\perp\times\mathbf{v} = \mathbf{E}\cdot\mathbf{B}\times\mathbf{v}$. Similarly, $\mathbf{E}_\perp\cdot\mathbf{v}\times\mathbf{B} = \mathbf{E}\cdot\mathbf{v}\times\mathbf{B}$ and we see that these two terms cancel. Also $|\mathbf{v}\times\mathbf{E}|^2 = v^2E^2_\perp$ and $|\mathbf{v}\times\mathbf{B}|^2 = v^2B^2_\perp$, so

$$
B'^2 - E'^2/c^2 = \gamma^2\left[B^2_\perp(1-\beta^2) - \frac{E^2_\perp}{c^2}(1-\beta^2)\right] + B^2_\parallel - \frac{E^2_\parallel}{c^2}.
$$

Finally, $\gamma^2(1-\beta^2) = 1$ so the rhs is $B^2_\perp + B^2_\parallel - (E^2_\perp + E^2_\parallel)/c^2$ as required.

4. *Show that with $S_{\mu\nu} = u_\mu v_\nu - u_\nu v_\mu$, Tr$\mathbf{S}\cdot\overline{\mathbf{S}} = 0$. This result explains why* $\mathbf{S}$ *has only 5 degrees of freedom (Exercise 4).*

Let's make a general transformation

$$\mathbf{u} \to \mathbf{u}' = a\mathbf{u} + b\mathbf{v} \qquad \mathbf{v} \to \mathbf{v}' = c\mathbf{u} + d\mathbf{v}.$$

Then

$$\begin{aligned} S'_{\mu\nu} &= (u'_\mu v'_\nu - u'_\nu v'_\mu) = (au_\mu + bv_\mu)(cu_\nu + dv_\nu) - (au_\nu + bv_\nu)(cu_\mu + dv_\mu) \\ &= (ad - bc)u_\mu v_\nu + (bc - da)u_\nu v_\mu = (ad - bc)S_{\mu\nu}. \end{aligned}$$

Thus $\mathbf{S}' = \mathbf{S}$ providing $ad - bc = 1$: the invariance of $\mathbf{S}$ implies one constraint on four numbers and so three are free. Thus only $5 = (8 - 3)$ numbers are required to specify $\mathbf{S}$. How do we reconcile this with the fact that $\mathbf{S}$ has six non-zero entries? Consider

$$S_{\mu\nu}\overline{S}^{\mu\nu} = \tfrac{1}{2}S_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}S_{\alpha\beta} = 2u_\mu v_\nu u_\alpha v_\beta \epsilon^{\mu\nu\alpha\beta}.$$

This vanishes because the product $u_\mu u_\alpha$ is symmetric in $\mu\alpha$, while the L-C symbol is antisymmetric in these indices. So the six numbers in $\mathbf{S}$ are not in fact free – they satisfy one constraint, making only five of them free.

5. *Relate the above statements to the number of independent components of an antisymmetric $n \times n$ matrix for $n = 2, 3, 4$.*

There are $\frac{1}{2}n(n-1)$ independent elements of an $n \times n$ antisymmetric matrix, so there are 1, 3, 6 independent elements for n = 2, 3, 4. In 2d an area has only a magnitude. In 3d it has magnitude and direction. In 4d it has magnitude and 4 angles.

6. Solution given in 4.

7. *Show that a uniform magnetic field parallel to the z-axis is associated with tension (negative pressure) along the axis, and pressure in the perpendicular directions.*

If $\mathbf{B} = (0, 0, B)$ and $\mathbf{E} = 0$,

$$P_{ij} = \frac{1}{\mu_0}\left(\tfrac{1}{2}\delta_{ij}B^2 - B^2\delta_{i3}\delta_{j3}\right) = \frac{1}{2\mu_0}\begin{pmatrix} B^2 & & \\ & B^2 & \\ & & -B^2 \end{pmatrix}$$

Thus there is pressure in the x, y directions and tension in the z direction.

8. *Show that when λ, μ and ν equal 1, 2 and 3 respectively, (1.48) becomes $\nabla\cdot\boldsymbol{B} = 0$.*
(ii) Show that with equation (1.22) equation (1.48) may also be written $\overline{F}^{\mu\nu}{}_{,\nu} = 0$.

$F_{23,1} + F_{31,2} + F_{12,3} = B_{x,x} + B_{y,y} + B_{z,z} = \nabla\cdot\mathbf{B}$

$$\overline{F}^{\mu\nu}{}_{,\nu} = \begin{pmatrix} B_{x,x} + B_{y,y} + B_{z,z} \\ -\partial_t B_x/c - \partial_y E_z/c + \partial_z E_z/c \\ -\partial_t B_y/c + \partial_x E_z/c - \partial_z E_x/c \\ -\partial_t B_z/c - \partial_x E_y/c + \partial_y E_x/c \end{pmatrix} = \begin{pmatrix} \nabla\cdot\mathbf{B} \\ \dfrac{1}{c}(\partial_t\mathbf{B} + \nabla\times\mathbf{E}) \end{pmatrix}$$

For comparison (1.48) with $\lambda, \mu, \nu = 0, 2, 3$ is

$$F_{23,0} + F_{30,2} + F_{02,3} = \frac{1}{c}\left(\partial_t B_x + \partial_y E_z - \partial_z E_y\right) = \frac{1}{c}(\partial_t\mathbf{B} + \nabla\times\mathbf{E})_x$$