Classical Fields III

1. The standard covariant derivative, $\nabla_\mu p^\nu = \partial_\mu p^\nu + \Gamma^\nu_{\lambda\mu} p^\lambda$, acts on 4-vectors that inhabit the four-dimensional "tangent space" of the space-time manifold. In particle physics other vector spaces are associated with each event. For example, a complex scalar field ψ associates with each event $\mathbf{x}$ a point in the complex plane – a two-dimensional vector space. Let e_1 and e_2 be two unimodular complex numbers. Show that we can write $\psi = \psi^1 e_1 + \psi^2 e_2$, where the ψ^a are real numbers.

If we make a different choice of basis numbers e_a at each event $\mathbf{x}$, $\partial_\mu \psi^a$ will not vanish even if ψ is the same everywhere. To detect this hidden equality we define a connection

$$D_\mu \psi^a = \partial_\mu \psi^a + \Gamma^a_{b\mu} \psi^b,$$

where $\boldsymbol{\Gamma}_\mu$ is a 2×2 matrix.

In quantum mechanics an e.m. field affects the dynamics through the replacement of the usual momentum operator by $p_\mu = -\mathrm{i}\hbar\{\partial_\mu - \mathrm{i}(q/\hbar)A_\mu\}$. Show that for an appropriate choice of $\boldsymbol{\Gamma}_\mu$ this can be written $p_\mu = -\mathrm{i}\hbar D_\mu$.

2. The curvature tensor is most conveniently defined by $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) Z^\alpha = R^\alpha{}_{\beta\mu\nu} Z^\beta$, which holds for any field $\mathbf{Z}$. From this definition derive an expression for $\mathbf{R}$ in terms of the Christoffel symbols.

In the notation of the previous problem, we define the curvature tensor for a scalar complex field through $(D_\mu D_\nu - D_\nu D_\mu)\psi^a = R^a{}_{b\mu\nu}\psi^b$. Assume that, as in the previous problem, summation over the index of ψ can be absorbed into complex multiplication, so we can write simply $(D_\mu D_\nu - D_\nu D_\mu)\psi = R_{\mu\nu}\psi$. Show that $R_{\mu\nu} = -\mathrm{i}(q/\hbar)F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field tensor.

3. With coordinates $x^\mu = (t, r, \theta, \phi)$ the Schwarzschild metric may be written

$$g_{\mu\nu} = \begin{pmatrix} -c^2 D & 0 & 0 & 0 \\ 0 & D^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \quad \text{where} \quad \begin{cases} D \equiv 1 - \dfrac{r_s}{r}, \\ r_s \equiv 2GM/c^2. \end{cases}$$

Show that the only non-vanishing Christoffel symbols of the form $\Gamma^t_{\mu\nu}$ are

$$\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{D'}{2D}.$$

From the equation of motion of a photon of momentum $\hbar\mathbf{k}$, show that in the Schwarzschild metric the time component $\omega \equiv k^0$ of a photon's 4-vector obeys

$$\frac{\mathrm{d}(\omega D)}{\mathrm{d}s} = 0 \quad \text{where the photon's path } x^\mu(s) \text{ satisfies} \quad k^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s},$$

and give a physical interpretation of this equation.

4. Derive the form of the energy-momentum tensor associated with a uniform magnetic field of strength B parallel to the x-axis. In which direction or directions does the field exert pressure?

5. A rope made of nylon of density ρ and cross-section A lies along the x-axis under tension F. Write down the form of the energy-momentum tensor inside the rope. Show that requiring that the energy density in the rope be positive for all observers, limits the permissible tension F.

6. A metric for the interior of a cosmic string is

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + r_0^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \mathrm{d}z^2,$$

where r_0 is a constant. Show that the only non-vanishing Christoffel symbols are

$$\Gamma^\theta_{\phi\phi} = -\tfrac{1}{2}\sin 2\theta \text{ and } \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.$$

Given that the only non-vanishing components of the Ricci tensor are $R_{\theta\theta}$ and $R_{\phi\phi}$ and that the edge of the string is at $\theta = \theta_m$, show that the tension in the string is $F = c^4(1 - \cos\theta_m)/(4G)$.

7. With (t, x, y, z) having their usual meanings, double-null coordinates for space-time are defined by

$$u = ct - x \quad y' = y$$
$$v = ct + x \quad z' = z \ .$$

Write down the Minkowski line element in double-null coordinates.

Consider the line element

$$\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v + f^2\mathrm{d}y^2 + g^2\mathrm{d}z^2,$$

where $f(u)$ and $g(u)$. Show that the only non-vanishing Cristoffel symbols are

$$\Gamma^v_{yy} = 2ff' \ , \ \Gamma^v_{zz} = 2gg' \ , \ \Gamma^y_{yu} = \Gamma^y_{uy} = f'/f \ , \ \Gamma^z_{zu} = \Gamma^z_{uz} = g'/g \ .$$

Hence, or otherwise, show that trajectories on which the spatial coordinates x, y, z are constant are geodesics.

The metric's Ricci tensor vanishes provided

$$\frac{f''}{f} + \frac{g''}{g} = 0,$$

where a prime denotes differentiation with respect to u. Show that this equation is satisfied by the choice

$$f(u) = 1 + \frac{u}{L}\Theta(u) \ , \ g(u) = 1 - \frac{u}{L}\Theta(u),$$

where L is a constant and $\Theta(u)$ is the Heaviside step function that vanishes for $u < 0$ and is unity for $u > 0$.

For the above choice of f and g, determine as a function of time the invariant distance between particles that move on $x = 0$, $y = 0$, $z = \pm a$, and similarly the distance between particles that move on $x = 0$, $y = \pm a$, $z = 0$.

Interpret your results physically.

8. The Robertson-Walker metric may be written

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right].$$

Explain the significance of the quantities a and K, and of the world-lines $(r, \theta, \phi) =$ constant.

Show that photons can travel down curves ($\theta =$ constant, $\phi =$ constant).

Given that $a = (t/t_0)^{2/3}$ for $K = 0$, find the distance now (t_0) in the case $K = 0$ between us and a galaxy from which we are currently receiving photons emitted at t_1.

Suppose the Universe is closed with the Earth at the point $r = 0$. A distant galaxy of radius R is currently distance D from us with its centre on the line $\theta = 0$. Show that its rim is at angular coordinate

$$\theta = \frac{(1 + z)R\sqrt{K}}{\sin(D\sqrt{K})}.$$

where z is the galaxy's redshift. Simplify this formula for the case $z \ll 1$ and discuss the difference between the general result and this case.

9. Show that for any two vectors **u**, **v** we have

$$(u^\alpha\nabla_\alpha)v^\beta - (v^\alpha\nabla_\alpha)u^\beta = [u, v]^\beta,$$

where the vector $[\mathbf{u}, \mathbf{v}]$ is defined by

$$[u, v]^\beta \equiv u^\alpha\partial_\alpha v^\beta - v^\alpha\partial_\alpha u^\beta.$$

For each fixed ϵ, $x^\alpha(\tau, \epsilon)$ defines a geodesic, with τ the affine parameter. Show that

$$\left[\frac{\mathrm{d}x}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\epsilon}\right]^\beta = 0.$$

Show further that

$$(\dot{x}^\alpha \nabla_\alpha)(\dot{x}^\beta \nabla_\beta)\frac{\mathrm{d}x^\gamma}{\mathrm{d}\epsilon} = R^\gamma{}_{\lambda\mu\nu}\dot{x}^\lambda \dot{x}^\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\epsilon},$$

where $\dot{\mathbf{x}} \equiv \mathrm{d}\mathbf{x}/\mathrm{d}\tau$ and the curvature tensor $\mathbf{R}$ can be taken to be defined by

$$\Big((u^\alpha \nabla_\alpha)(v^\beta \nabla_\beta) - (v^\alpha \nabla_\alpha)(u^\beta \nabla_\beta) - [u, v]^\alpha \nabla_\alpha\Big) w^\gamma = R^\gamma{}_{\lambda\mu\nu} u^\mu v^\nu w^\lambda,$$

with $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ arbitrary vectors.

Two masses are dropped from points a small height ϵ apart. Show that just after they are released, the separation $\delta\mathbf{x}$ between them satisfies

$$\frac{D^2 \delta x^\gamma}{Dt^2} = c^2 R^\gamma{}_{00\nu}\delta x^\nu,$$

where $x^0 \equiv ct$. Hence show that the gravitational field at the Earth's surface has the curvature component

$$R^z{}_{00z} = 2g/(c^2 R)$$

where z is an upwards directed coordinate, g is the usual acceleration due to gravity, and R is the Earth's radius.