Classical Fields II

1. Show that the affine distance

$$s \equiv \int_a^b \sqrt{\left|g_{\mu\nu}\frac{\mathrm{d}x'^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x'^\nu}{\mathrm{d}\lambda}\right|}\,\mathrm{d}\lambda.$$

is extremal along curves x^τ that satisfy

$$\frac{\mathrm{d}^2x'^\mu}{\mathrm{d}s^2} + \Gamma'^\mu_{\kappa\alpha}\frac{\mathrm{d}x'^\kappa}{\mathrm{d}s}\frac{\mathrm{d}x'^\alpha}{\mathrm{d}s} = 0.$$

2. Prove that $A'_{\mu;\nu} - A'_{\nu;\mu} = A'_{\mu},_{\nu} - A'_{\nu},_{\mu}$.

3. Show that $\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\rho}\partial_\nu g_{\rho\mu}$. Show also that for any infinitesimal change $\delta\mathbf{A}$ in a matrix $\mathbf{A}$ we have $\delta\ln(\det\mathbf{A}) = \mathrm{Tr}(\mathbf{A}^{-1}\delta\mathbf{A})$. Hence show that

$$\Gamma^\mu_{\mu\nu} = \partial_\nu\ln(\sqrt{g},) \qquad (\dagger)$$

where $g \equiv |\det g|$.

4. For a weak gravitational field we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where all components of $\mathbf{h}$ are small. Show that under an infinitesimal coordinate change $\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \boldsymbol{\xi}$, to first order in small quantities we have

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu.$$

Explain by analogy with electromagnetism why this can be considered a gauge transfomation.

5. Show that under a gauge transformation $\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \boldsymbol{\xi}$, the Christoffel symbol transforms as

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} - \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\rho x^\sigma}.$$

Hence show that we can always choose a gauge in which the **harmonic gauge** condition

$$\Gamma^\lambda \equiv g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$$

is satisfied.

Use the result (†) above to show that the harmonic gauge condition can be writen

$$0 = \partial_\kappa(\sqrt{g}\,g^{\lambda\kappa})\,.$$

Hence show that when the harmonic gauge condition holds, the covariant d'Alembertian

$$\Box\phi \equiv \nabla_\kappa(g^{\kappa\lambda}\nabla_\lambda\phi)$$

is simply

$$\Box\phi = g^{\lambda\kappa}\frac{\partial^2\phi}{\partial x^\lambda\partial x^\kappa}.$$

Thus when the harmonic gauge condition is satisfied, each coordinate is a harmonic function: $\Box x^\alpha = 0$.

6. Show that

$$g_{\lambda\sigma}\partial_\kappa g^{\sigma\rho} = -g^{\sigma\rho}\big(\Gamma^\eta_{\kappa\lambda}g_{\eta\sigma} + \Gamma^\eta_{\kappa\sigma}g_{\eta\lambda}\big).$$

Hence, or otherwise, show that the curvature tensor can be written

$$R_{\lambda\mu\nu\kappa} = \tfrac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right) + g_{\eta\sigma}\big(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\big).$$

Deduce that $R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0$. Hence show that $R_{\lambda\mu\nu\kappa}$ has 20 independent indices.

7. Show that for a weak field ($\mathbf{g} = \boldsymbol{\eta} + \mathbf{h}$), to first order in $\mathbf{h}$ the Ricci tensor is

$$R_{\alpha\beta} = \tfrac{1}{2}\partial_\beta\partial_\alpha h - \tfrac{1}{2}\partial^\nu(\partial_\beta h_{\nu\alpha} + \partial_\alpha h_{\nu\beta} - \partial_\nu h_{\alpha\beta}),$$

where $h = h^\alpha_\alpha$. Show further that to first order the harmonic gauge condition reads

$$0 = 2\partial^\mu h_{\lambda\mu} - \partial_\lambda h.$$

Hence show that, in the harmonic gauge and to first order, Einstein's equations, $R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = -(8\pi G/c^4)T_{\alpha\beta}$, read

$$\Box\overline{h}_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}, \qquad (\ddagger)$$

where $\overline{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta}$.

Write down an analogous equation of electromagnetism and explain the physical significance of this result.

8. Use equation (‡) above to show that in harmonic coordinates the metric associated with a weak gravitational field that is caused by a static distribution of rest-mass takes the form

$$\mathrm{d}s^2 = -\Big(1 + 2\frac{\Phi}{c^2}\Big)c^2\mathrm{d}t^2 + \Big(1 - 2\frac{\Phi}{c^2}\Big)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2),$$

where Φ should be related to T_{00}.