Classical Fields I

1. A π^0 meson moving with velocity $\boldsymbol{v}$ decays into two gamma-rays. If α denotes the angle between the directions of travel of the two photons, express $\cos\alpha$ in terms of $\beta \equiv v/c$ and $\cos\theta'$, where θ' is the direction of emission of one of the photons in the centre of mass system. Hence show that the angular correlation is given by

$$\mathrm{d}N = \frac{\sin\alpha\,\mathrm{d}\alpha}{4\beta\gamma^2 \sin^3 \frac{1}{2}\alpha\sqrt{\beta^2 - \cos^2 \frac{1}{2}\alpha}}.$$

[The π^0 meson has zero spin.]

2. In a certain frame of reference S, a hollow non-magnetic conductor moves with uniform velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$. Find in the frame S the electric and magnetic fields in the space within the conductor.

3. A certain quantity X is known to be a Lorentz scalar and to depend only on the momenta $\boldsymbol{p}_a$, $\boldsymbol{p}_b$ of two particles and on the electric and magnetic fields. In a constant magnetic field $\boldsymbol{B}$ and zero electric field, X is known to take the form $(\boldsymbol{p}_a \times \boldsymbol{p}_b) \cdot \boldsymbol{B}$. Find the form taken by X when a constant electric field is also present.

4. Show that the total e.m. force $\boldsymbol{f}$ acting on a localized distribution of charge and current satisfies

$$\boldsymbol{f} + \frac{\mathrm{d}}{\mathrm{d}t}\int (\boldsymbol{D} \times \boldsymbol{B})\,\mathrm{d}^3\boldsymbol{r} = \int \boldsymbol{W}\,\mathrm{d}^3\boldsymbol{r},$$

where $\boldsymbol{W} \equiv \boldsymbol{E}\nabla \cdot \boldsymbol{D} - \boldsymbol{B} \times (\nabla \times \boldsymbol{H}) - \boldsymbol{D} \times (\nabla \times \boldsymbol{E})$ and the integration is over the region within which the charges and currents are confined. Show that in the vacuum $\boldsymbol{n} \cdot \boldsymbol{W} = \nabla \cdot \boldsymbol{U}(\boldsymbol{n})$, where

$$\boldsymbol{U}(\boldsymbol{n}) \equiv \epsilon_0\big[(\boldsymbol{n}\cdot\boldsymbol{E})\boldsymbol{E} - \tfrac{1}{2}\boldsymbol{n}E^2\big] + \frac{1}{\mu_0}\big[(\boldsymbol{n}\cdot\boldsymbol{B})\boldsymbol{B} - \tfrac{1}{2}\boldsymbol{n}B^2\big]$$

and $\boldsymbol{n}$ is any constant vector.

Explain the physical meaning of $\boldsymbol{D} \times \boldsymbol{B}$ and $\boldsymbol{U}(\boldsymbol{n})$ and the relation of $\boldsymbol{U}$ to the relativistic energy-momentum tensor $\mathbf{T}$.
[$\boldsymbol{a} \times (\nabla \times \boldsymbol{a}) = \frac{1}{2}\nabla(a^2) - (\boldsymbol{a}\cdot\nabla)\boldsymbol{a}$, where $\boldsymbol{a}$ is any vector field.]

5. The components of a 4-vector $\mathbf{v}$ are arranged to make a 2×2 Hermitian matrix

$$\mathbf{X} = \begin{pmatrix} v_0 + v_z & v_x - \mathrm{i}v_y \\ v_x + \mathrm{i}v_y & v_0 - v_z \end{pmatrix}.$$

Show that for given $\mathbf{X}$ it is in general impossible to find a Pauli spinor $\boldsymbol{\eta}$ such that $X_{ij} = \eta_i\eta_j^*$.

6. If we complement the usual Pauli matrices by $\sigma_0 \equiv I$, the identity matrix, show that the rule for rotating a four-vector x^μ can be written

$$\mathrm{e}^{\mathrm{i}\theta\sigma_{\boldsymbol{n}}} x^\mu \sigma_\mu \mathrm{e}^{-\mathrm{i}\theta\sigma_{\mathbf{n}}} = x'^\mu\sigma_\mu,$$

where $\sigma_{\boldsymbol{n}} \equiv \boldsymbol{n}\cdot\boldsymbol{\sigma}$. Show that any vector that is invariant under this transformation has a spatial part that is proportional to $\boldsymbol{n}$, and explain the physical significance of this result.

7. ϕ is a real scalar field and $V(\phi)$ is a potential function. Derive a field equation for ϕ from

$$\mathcal{L} = -\left[\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi)\right]. \tag{1}$$

Explain the connection of this equation to the Klein–Gordon equation. Taking $V = 1 - \cos(\phi)$ derive the Sine–Gordon equation. If $\phi(x,t) = \Phi(x - \beta ct)$, which β a constant, show that Φ satisfies

$$\tfrac{1}{2}(1-\beta^2)(\Phi')^2 - 2\sin^2\tfrac{1}{2}\Phi = A,$$

where a prime denotes differentiation and A is a constant. Setting $A = 0$ and taking $\beta^2 < 1$ obtain the soliton solution

$$\tan\tfrac{1}{4}\Phi = \pm\exp\left[\pm\gamma(X - X_0)\right],$$

where $\gamma = 1/\sqrt{1-\beta^2}$ and $X = x - \beta ct$. Describe the solution obtained when both signs are positive.

8. Show that the energy density of the field ϕ that is governed by (1) is

$$T^{00} = \tfrac{1}{2}[(\partial_0\phi)^2 + |\nabla\phi|^2] + V(\phi).$$

9. ψ is a Dirac-spinor field. Show that

$$\mathcal{L} = \overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi$$

is invariant under a global phase shift

$$\psi \to \mathrm{e}^{\mathrm{i}\theta}\psi.$$

Show that this invariance is associated with the conserved current

$$j^\mu = \mathrm{i}\overline{\psi}\gamma^\mu\psi.$$

10. Defining $\boldsymbol{\psi} = \boldsymbol{E} + \mathrm{i}c\boldsymbol{B}$ show that Maxwell's eqns for source-free fields can be written

$$\frac{\partial\boldsymbol{\psi}}{\partial t} + \mathrm{i}c\nabla\times\boldsymbol{\psi} = 0 \quad ; \quad \nabla\cdot\boldsymbol{\psi} = 0. \tag{†}$$

Show that the e.m. energy density is $\frac{1}{2}\epsilon_0\boldsymbol{\psi}^*\cdot\boldsymbol{\psi}$ and the Poynting vector is $\dfrac{1}{2\mathrm{i}c\mu_0}\boldsymbol{\psi}^*\times\boldsymbol{\psi}$. Show further that the field

$$\boldsymbol{\psi} = \begin{pmatrix}1\\ \mathrm{i}\\ 0\end{pmatrix}\mathrm{e}^{\pm\mathrm{i}(kz-\omega t)} \tag{†}$$

with $\omega = kc$ represents a circularly polarized plane wave. Discuss the parallel between equation (†) and the Dirac equation for the electron wave-function.