University of Antwerp Autumn 2016

Introduction to Quantum Chromodynamics

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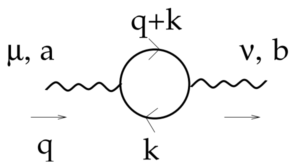
Problem Set 2

Problem 1

The fermion loop contribution to the gluon self-energy is given by the graph



i) Write down the one-loop integral for $i\pi^{ab}_{\mu\nu}(q)$.

ii) Work in dimensional regularization with $d = 4 - 2\varepsilon$ space-time dimensions. Show that

$$\pi^{ab}_{\mu\nu}(q) = \left(g_{\mu\nu} q^2 - q_\mu q_\nu \right) \Pi(q^2) \ \ ,$$

where

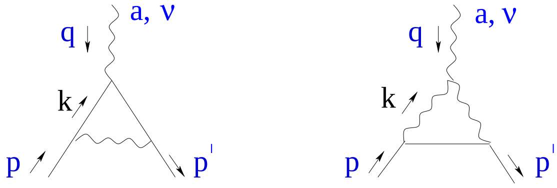
$$\Pi(q^2) = -\mathrm{Tr}(T^a T^b) \ \frac{g^2}{4\pi^2} \ \Gamma(\varepsilon) \int_0^1 dx \ \left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2} \right)^\varepsilon 2x(1-x) \ \ .$$

iii) Show that the ultraviolet divergent part of Π is given by

$$-\mathrm{Tr}(T^a T^b) \ \frac{\alpha_s}{3\pi} \ \frac{1}{\varepsilon} \ \ .$$

Problem 2

The renormalization of the quark-quark-gluon vertex at one loop is given by the following graphs



(a) (b)

i) Show that the color factors for graphs (a) and (b) are given by

$$T^c T^a T^b \delta^{bc} = (C_F - \frac{1}{2} C_A) T^a \qquad \text{(a)} \quad , \qquad T^b T^c f^{abc} = \frac{i}{2} C_A T^a \qquad \text{(b)} \quad .$$

ii) Work in dimensional regularization with $d = 4 - 2\varepsilon$ space-time dimensions. Show that the ultraviolet-divergent parts of graphs (a) and (b) are given in Feynman gauge by

$$\text{graph (a)} = \frac{ig^3}{(4\pi)^2} (C_F - \frac{1}{2} C_A) T^a \gamma^\nu \frac{1}{\varepsilon} + \ldots \quad , \qquad \text{graph (b)} = \frac{ig^3}{(4\pi)^2} \frac{3}{2} C_A T^a \gamma^\nu \frac{1}{\varepsilon} + \ldots$$

iii) Combining the results above for graphs (a) and (b), determine the vertex renormalization constant Z_1:

$$Z_1 = 1 - \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} (C_F + C_A) \quad .$$

iv) Discuss how the above result combines with the following results for the quark and gluon wave-function renormalization constants Z_2 and Z_3,

$$Z_2 = 1 - \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} C_F \quad , \qquad Z_3 = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} (\frac{5}{3} C_A - \frac{4}{3} N_f T_F) \quad ,$$

to give the β function at one loop

$$\beta(\alpha_s) = -\frac{1}{12\pi} (11 N_c - 2 N_f) \, \alpha_s^2 + \mathcal{O}(\alpha_s^3) \quad .$$

Problem 3

Consider the axial currents of QCD with two species of quarks,

$$j_A^{\mu k} = \bar{\psi} \gamma^\mu \gamma^5 \frac{\sigma^k}{2} \psi \quad , \quad j_A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi \quad ,$$

where

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \quad .$$

i) Write the chiral anomaly group-theory factors, and show that the isosinglet current j_A^μ has an anomaly from QCD interactions while the isotriplet current $j_A^{\mu k}$ does not. ii) By the same method show that the current $j_A^{\mu k}$ has an anomaly from quarks' electromagnetic interactions.

Problem 4

The perturbation expansion of the cross section σ for e^+e^- annihilation into hadrons may be written as a function of the total momentum square Q^2, the renormalization scale μ and the coupling α_s at scale μ, as follows

$$\begin{aligned} \sigma(Q^2, \alpha_s(\mu), \mu/Q) &= \sigma_0(Q^2) \left\{1 + [c_1 + c_1' \ln(\mu^2/Q^2)]\alpha_s(\mu) \right. \\ &+ \left. [c_2 + c_2' \ln(\mu^2/Q^2) + c_2'' \ln^2(\mu^2/Q^2)]\alpha_s^2(\mu) + \mathcal{O}(\alpha_s^3)\right\} \quad , \end{aligned}$$

where the lowest-order contribution σ_0 from the electromagnetic coupling of N_f species of quarks is given by $\sigma_0(Q^2) = 4\pi\alpha^2 N_c(\sum_f e_f^2)/(3Q^2)$, and the c's are perturbatively-calculable numerical coefficients.

Discuss the conditions on the higher-order coefficients imposed by renormalization group invariance of the cross section. Determine the value of c_2''. Determine the relation between c_2' and c_1. [Consider a renormalization group transformation $\mu \to Q$, $\alpha_s(\mu) \to \alpha_s(Q)$. Use the beta function to relate the values of the coupling at mass scales μ and Q, order-by-order.]

Problem 5

Consider DGLAP evolution equations for the parton distribution functions $f_a(x, \mu^2)$,

$$\frac{\partial\, f_a(x, \mu^2)}{\partial \ln \mu^2} = \sum_b \int_x^1 \frac{dz}{z}\, P_{ab}(\alpha_s(\mu^2), z)\, f_b(x/z, \mu^2) \quad ,$$

where the splitting functions P_{ab} are given as power series expansions in α_s.

i) Discuss the solution of the DGLAP equations at leading order in α_s in the limit $x \to 1$. Show that solutions are dominated by flavor-conserving evolution and can be approximated by

$$f_a(x, \mu^2) \simeq f_a(x, \mu_0^2) \exp\left\{\frac{C_a}{\pi\beta_0} \ln(1-x) \ln[\alpha_s(\mu_0^2)/\alpha_s(\mu^2)]\right\} \quad , \qquad x \to 1 \; ,$$

where $\beta_0 = (11N_c - 2N_f)/(12\pi)$, $C_a = C_F$ for $a = q$ and $C_a = C_A$ for $a = g$.

ii) Discuss the solution of the DGLAP equations at leading order in α_s in the limit $x \to 0$. Show that this limit is dominated by gluon evolution, and the solution can be approximated for small x and large μ^2 by

$$f_g(x, \mu^2) \simeq f_g(x, \mu_0^2) \exp\left\{2\sqrt{\frac{C_A}{\pi\beta_0} \ln(1/x) \ln[\alpha_s(\mu_0^2)/\alpha_s(\mu^2)]}\right\} \quad , \qquad x \to 0 \; ,$$

which grows faster than any power of $\ln 1/x$ for $x \to 0$ but slower than any power of $1/x$.