University of Antwerp Autumn 2016

Introduction to Quantum Chromodynamics

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Problem Set 1

Problem 1

i) Consider the amplitude for propagation of a free particle from point $\mathbf{x}$ to point $\mathbf{y}$ in quantum mechanics,

$$A(t) = \langle \mathbf{y} | e^{-iHt/\hbar} | \mathbf{x} \rangle \ .$$

Evaluate the amplitude using $H = \mathbf{p}^2/(2m)$. Discuss the implications of the result in terms of causality.

ii) Illustrate how the above result changes by using the relativistic expression for energy $E = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}$ in the amplitude $A(t)$.

iii) Now consider the theory of a quantum field

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a^\dagger_{\mathbf{p}} e^{ipx} \right) \ ,$$

where $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ are the annihilation and creation operators. Show that the commutator of two fields vanishes at space-like separations,

$$[\phi(x), \phi(y)] = 0 \ , \quad (x-y)^2 < 0 \ .$$

Interpret this result in terms of causality and propagation of particles and antiparticles.

Problem 2

i) Show that Maxwell's equations of electromagnetism may be obtained as Euler-Lagrange equations from the Lagrangian

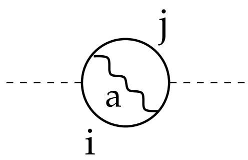
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \ .$$

ii) Write down the gauge transformations of electromagnetism in this formalism, and discuss the meaning of the condition $\partial^\mu J_\mu = 0$.

iii) Consider the Fourier components $\widetilde{A}^\mu(k)$ of the electromagnetic field as functions of the wave vector k. Decompose $\widetilde{A}^\mu(k)$ along a a basis of four polarization vectors, with two of them being transversal to k. Use the gauge transformations to show that two of these polarizations are redundant and that only the two transversal polarizations are physical degrees of freedom.

Problem 3

i) Compute the color factor for the diagram



where solid lines are quarks, wavy lines are gluons and dashed lines are colorless particles. [Answ.: $C_F N_c = (N_c^2 - 1)/2$.]

ii) How does the annihilation cross section $\sigma(e^+e^- \to \text{hadrons})$ scale with the number of colors N_c? And the Drell-Yan lepton pair production cross section? And the branching ratio $B(W^- \to e^- \,\overline{\nu})$?

Problem 4

i) Show that the QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_f \overline{\psi}_f (i\not{D} - m_f)\psi_f$$

is invariant under the nonabelian gauge transformations

$$\psi \to G\psi \quad , \qquad A_\mu \to G A_\mu G^{-1} - \frac{i}{g}(\partial_\mu G) G^{-1} \quad ,$$

where $G = \exp(i\alpha^a T^a)$, $A_\mu \equiv A^a_\mu T^a$, and T^a are the color-charge matrices obeying $[T^a, T^b] = i f^{abc} T^c$.

ii) Show that for small angles $\alpha^a \ll 1$ the gauge transformations on A^a_μ have the form

$$A^a_\mu \to A^a_\mu + \frac{1}{g}\, \partial_\mu \alpha^a - f^{abc} \alpha^b A^c_\mu \,.$$

iii) Show that the infinitesimal gauge transformations in part ii) can be rewritten compactly in the form

$$\delta A^a_\mu = \frac{1}{g}\, D^{ac}_\mu \alpha^c \,,$$

where D_μ^{ac} is the covariant derivative in the adjoint representation, and identify the expression for D_μ^{ac}.

Problem 5

The equation for the gluon propagator $D_{\nu\rho}^{ab}(x)$ is given, upon including the gauge-fixing term $-(2\xi)^{-1}(\partial^\mu A_\mu^a)^2$ in the Lagrangian, by

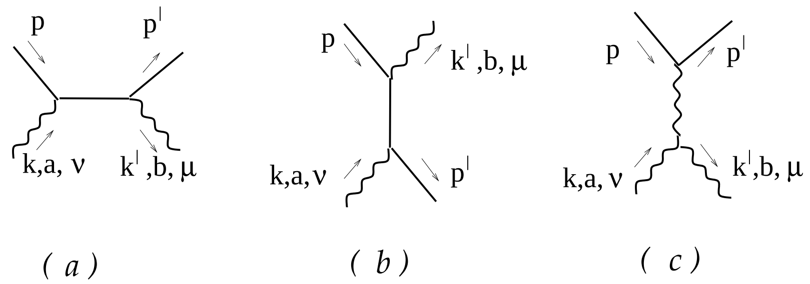
$$\left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu + \frac{1}{\xi}\, \partial^\mu \partial^\nu\right) D_{\nu\rho}^{ab}(x) = i\delta_\rho^\mu \delta^4(x)\delta^{ab} \quad .$$

By Fourier-transforming this equation and applying Green's function techniques, show that the gluon propagator in momentum space is given for this class of gauge choices by

$$\widetilde{D}_{\mu\nu}^{ab}(k) = \frac{-i\delta^{ab}}{k^2 + i\varepsilon} \left[g_{\mu\nu} - (1-\xi)\, \frac{k_\mu k_\nu}{k^2} \right] \quad .$$

Problem 6

Quark-gluon Compton scattering is given, at the lowest order in the QCD coupling g, by the graphs



(i) Show that the sum of graphs (a) and (b) dotted into k^ν gives

$$M_{\mu\nu}^{(a)+(b)}\, k^\nu = ig^2 [T^a, T^b] \overline{u}(p')\gamma_\mu u(p) \; .$$

(ii) Evaluate the contribution of graph (c) dotted into k^ν,

$$M_{\mu\nu}^{(c)}\, k^\nu \; ,$$

and show that the sum of all graphs gives $M_{\mu\nu} k^\nu = 0$ provided μ is restricted to physical polarizations. Contrast this with the abelian case of electrodynamics. Discuss implications of these results in terms of longitudinal polarization states in the photon and gluon cases.