Introduction to QED and QCD

Problem sheet 3

Problem 1

Consider the scattering process in Fig. 1 and show that the initial-state flux factor defined as

$$\mathcal{J} = 4E_a E_b |v_a - v_b|$$

is relativistically invariant and equals

$$\mathcal{J} = 4|p_i^{(c.m.)}|\sqrt{s} = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} \ ,$$

where p_a, p_b and m_a, m_b are the four-momenta and masses of particles a and b, $s = (p_a + p_b)^2$, and $p_i^{(c.m.)}$ is the initial three-momentum in the center of mass frame.

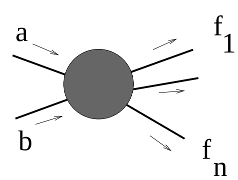

Figure 1: *Scattering process* $a + b \to f_1 + f_2 + \ldots f_n$.

Problem 2

Consider a scattering amplitude of the form $\overline{u}\ \Gamma\ v$, given by a generic product Γ of Dirac γ matrices sandwiched between u and v spinors. Show that the sum over all spins of the squared amplitude evaluates to

$$\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} |\overline{u}_\alpha(p')\Gamma v_\beta(p)|^2 = \mathrm{Tr}\left[\Gamma(\not{p} - m)\gamma^0\Gamma^\dagger\gamma^0(\not{p}' + m)\right] \ .$$

Problem 3

Using the result above, show that the invariant matrix element square for the annihilation process $e^+(k')e^-(k) \to \mu^+(p')\mu^-(p)$ in Fig. 2, averaged over initial spins and summed over final spins, is given by

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{4} \frac{e^4}{(q^2)^2} \, \text{Tr}\Big[\gamma^\rho \, (\not{p}' - M) \, \gamma^\tau \, (\not{p} + M)\Big] \, \text{Tr}\Big[\gamma_\tau \, \not{k}' \, \gamma_\rho \, \not{k}\Big] \quad ,$$

where we take the approximation of massless electrons.

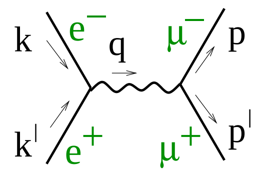

Figure 2: *Annihilation of electron pairs into muon pairs.*

Problem 4

Consider the center-of-mass reference frame for the process in Fig. 2. Show that the phase space element $d\Phi$ for the two-body final state can be written as

$$d\Phi = \frac{|\mathbf{p}|}{16\pi^2\sqrt{s}} \, d\Omega \quad ,$$

where $d\Omega = \sin\theta d\theta d\varphi$, $s = q^2$, and $|\mathbf{p}| = (\sqrt{s}/2)\sqrt{1 - 4M^2/s}$.

Problem 5

The computation of the traces in the matrix element square $\overline{|\mathcal{M}_{fi}|^2}$ above gives

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{8e^4}{s^2} \, \Big[(p \cdot k)^2 + (p \cdot k')^2 + M^2 \, k \cdot k'\Big] \quad .$$

Express this result in terms of the center-of-mass scattering angle θ and obtain that the differential cross section $d\sigma/d\Omega$ for $e^+e^- \to \mu^+\mu^-$ is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^-\to\mu^+\mu^-} = \frac{\alpha^2}{4 \, s} \sqrt{1 - \frac{4M^2}{s}} \, \left[1 + \frac{4M^2}{s} + \left(1 - \frac{4M^2}{s}\right) \cos^2\theta\right] \quad .$$

Problem 6

By integrating the differential cross section $d\sigma/d\Omega$ for $e^+e^- \to \mu^+\mu^-$ over angles, obtain that the total cross section is given in the high energy limit $4M^2/s \to 0$ by

$$\sigma_{\text{tot}} \simeq \frac{4 \, \pi \, \alpha^2}{3 \, s} \quad .$$