RAL HEP Summer School September 2011

Introduction to QED and QCD

Problem sheet 2

Problem 1

Given the algebra of rotation generators J_i and boost generators K_i,

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \ , \qquad [K_i, K_j] = -i\varepsilon_{ijk} J_k \ , \qquad [J_i, K_j] = i\varepsilon_{ijk} K_k \ ,$$

show that the hermitian generators

$$A_i = \frac{1}{2}(J_i + iK_i) \ , \qquad B_i = \frac{1}{2}(J_i - iK_i)$$

commute with one another and separately obey the commutation relations of angular momentum

$$[A_i, A_j] = i\varepsilon_{ijk} A_k \ , \qquad [B_i, B_j] = i\varepsilon_{ijk} B_k \ , \qquad [A_i, B_j] = 0 \ .$$

Problem 2

Let ξ be a Weyl spinor. Show that if ξ transforms in the $(\frac{1}{2}, 0)$ representation of the group of Lorentz transformations, then $\varepsilon\xi^*$, where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, transforms in the $(0, \frac{1}{2})$ representation. Write down a general expression for a Dirac spinor in terms of two left-handed Weyl spinors.

Problem 3

Verify the orthonormality and completeness relations for the solutions of the Dirac equation,

$$\bar{u}_r(\mathbf{p}) u_s(\mathbf{p}) = -\bar{v}_r(\mathbf{p}) v_s(\mathbf{p}) = 2m\delta^{rs} \ , \quad \bar{u}_r(\mathbf{p}) v_s(\mathbf{p}) = \bar{v}_r(\mathbf{p}) u_s(\mathbf{p}) = 0 \ ,$$

and

$$\sum_{r=1}^{2} u_r(\mathbf{p}) \bar{u}_r(\mathbf{p}) = (\not{p} + m) \ , \quad \sum_{r=1}^{2} v_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) = (\not{p} - m) \ .$$

Problem 4

Show that the Dirac hamiltonian H commutes with the total angular momentum operator

$$[\boldsymbol{L} + \mathbf{S}, H] = 0 \ ,$$

where $\boldsymbol{L}$ is the orbital angular momentum

$$\boldsymbol{L} = \boldsymbol{x} \wedge \boldsymbol{p}$$

and $\boldsymbol{S}$ is the spin operator

$$\boldsymbol{S} = \frac{1}{2}\,\boldsymbol{\Sigma} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \ .$$

Problem 5

Consider plane wave solutions of the Dirac equation

$$\psi_+(x) = \mathcal{N}\begin{pmatrix} \chi_r \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi_r \end{pmatrix} e^{-ipx} \equiv u_r(\mathbf{p})e^{-ipx} \quad , \quad \psi_-(x) = \mathcal{N}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi_r \\ \chi_r \end{pmatrix} e^{ipx} \equiv v_r(\mathbf{p})e^{ipx} \quad ,$$

where $\mathcal{N} = \sqrt{E+m}$, and the spinors χ_r for $r = 1, 2$ are given by

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ .$$

Take $\mathbf{p} = (0, 0, p^z)$. Verify that

$$S^z u_1 = \frac{1}{2}\ u_1 \quad , \quad S^z u_2 = -\frac{1}{2}\ u_2 \ ,$$

$$S^z v_1 = \frac{1}{2}\ v_1 \quad , \quad S^z v_2 = -\frac{1}{2}\ v_2 \ ,$$

where S^z is the z component of the spin operator.

Problem 6

Use the Dirac equation to derive the identity

$$\bar{u}(p')\gamma^\nu u(p) = \frac{1}{2m}\ \bar{u}(p')(p+p')^\nu u(p) + \frac{i}{m}\ \bar{u}(p')\ \Sigma^{\nu\rho}\ q_\rho\ u(p) \quad ,$$

where $q = p' - p$, $\Sigma^{\nu\rho} = (i/4)\ [\gamma^\nu, \gamma^\rho]$.