An Introduction to QED & QCD

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These lectures present a heuristic introduction to gauge theories of electromagnetic and strong interactions, focusing on perturbative applications of the S matrix and the use of Feynman graphs in QED and QCD. They are complementary to the presentations of field theory in the Standard Model and Quantum Field Theory courses at this School. The approach followed in these lectures may be found in the textbooks given in Ref. [1]. Quantum field theory treatments may be found in the textbooks given in Ref. [2].

Secs. 1 and 2 discuss relativistic quantum mechanics and spin. Secs. 3 to 5 are devoted to interactions and scattering processes at tree level in QED. Sec. 6 extends the discussion to QCD. Secs. 7 and 8 consider loops and give an introductory discussion to renormalization.

1 Relativistic quantum mechanics

In this section we present the Klein-Gordon equation and the Dirac equation as they arise from attempts to generalize quantum-mechanical wave equations to include relativity. We discuss the difficulties in interpreting these equations as single-particle wave equations, and illustrate that the Feynman-Stueckelberg causality argument points to the resolution of these difficulties by going beyond the single particle interpretation.

1.1 Relativistic wave equations

Let us start with the case of nonrelativistic quantum mechanics, and ask how we can generalize it to the relativistic case.

The time evolution of the state $|\psi\rangle$ of a quantum mechanical system is given by the Schrödinger equation,

$$i\,\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \ , \tag{1.1}$$

where H is the hamiltonian operator corresponding to the total energy.

For a free, spinless, nonrelativistic particle we have

$$H = \frac{\mathbf{p}^2}{2m} \ , \tag{1.2}$$

where $\mathbf{p}$ is the momentum operator, and m is the particle's mass. In the basis of eigenstates of the position operator, $\mathbf{p}$ is represented by

$$\mathbf{p} = -i\,\boldsymbol{\nabla} \ , \tag{1.3}$$

and therefore the evolution equation reads

$$i\,\frac{\partial}{\partial t}\psi(\mathbf{x},t) = -\frac{1}{2m}\,\boldsymbol{\nabla}^2\psi(\mathbf{x},t) \ , \tag{1.4}$$

where $\psi(\mathbf{x},t) = \langle\mathbf{x}|\psi\rangle$ is the position-space wave function.

The wave equation (1.4) admits a probabilistic interpretation. By taking the complex conjugate ψ^* times Eq. (1.4) and subtracting ψ times the complex conjugate equation, we obtain

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \ , \tag{1.5}$$

with

$$\rho = \psi^*\psi \ , \quad \mathbf{j} = -\frac{i}{2m}\,[\psi^*(\nabla\psi) - (\boldsymbol{\nabla}\psi^*)\psi] \ . \tag{1.6}$$

Here ρ and $\mathbf{j}$ are interpreted as probability density and current, and the continuity equation (1.5) expresses probability conservation.

How could we extend this to the relativistic case? To do this, we need to incorporate the relativistic energy-momentum relation

$$E^2 = \mathbf{p}^2 + m^2 \ . \tag{1.7}$$

If we naively were to take

$$H = \sqrt{\mathbf{p}^2 + m^2} \ , \tag{1.8}$$

this would yield the correct energy-momentum relation, but would give as a candidate wave equation

$$i \, \frac{\partial}{\partial t} \, \psi = \sqrt{\mathbf{p}^2 + m^2} \, \psi \ , \tag{1.9}$$

which contains space derivatives under the square root. This equation has a number of difficulties, because it treats time and space derivatives on a different footing, contrary to what one would expect of a relativistic theory, and because it is non-local in space, as the square root gives rise to an infinite number of spatial derivatives.

One possible way to overcome this is to square the differential operators in Eq. (1.9) before applying them to ψ. This gives

$$-\frac{\partial^2}{\partial t^2} \, \psi = (-\boldsymbol{\nabla}^2 + m^2) \, \psi \ , \tag{1.10}$$

that is, using covariant notation with $\partial^\mu = (\partial/\partial t, -\boldsymbol{\nabla})$, $\partial^2 = \partial^\mu \partial_\mu$,

$$(\partial^2 + m^2) \, \psi = 0 \ . \tag{1.11}$$

This is the approach originally proposed by Schrödinger, Klein and Gordon, and Eq. (1.11) is referred to as the Klein-Gordon equation. This equation describes relativistic spin-0 particles. We will discuss Klein-Gordon in Subsec. 1.2. As we will see, this equation is a candidate wave equation consistent with relativity, but it runs into problems with quantum mechanics as a single-particle wave equation, because, due to the second-order time derivative, it does not lead to a positive-definite probability density.

A second possible way around the naive Eq. (1.9) is to insist on the equation being first-order in the time derivative but devise a new hamiltonian H_d which is local, linear in momentum, and such that its square returns the correct relativistic energy-momentum relation (1.7):

$$i \, \frac{\partial}{\partial t} \, \psi = H_d \, \psi \ . \tag{1.12}$$

This is the approach followed originally by Dirac. It turns out that this route is viable only if the wave function is not one-component but multi-component (which implies spin), and the new hamiltonian is of the form

$$H_d = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m \ , \tag{1.13}$$

where α and β are four matrices, in a space to be determined, obeying the relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \ , \quad \beta \alpha_i + \alpha_i \beta = 0 \ , \quad \beta^2 = 1 \ . \tag{1.14}$$

Eq. (1.12), with H_d given in Eqs. (1.13),(1.14), is the Dirac equation. This equation describes relativistic spin-1/2 particles. We will discuss it in Subsec. 1.3.

We will see that, unlike the Klein-Gordon equation, the Dirac equation, being first-order, allows one to construct a positive-definite probability density. However, we will see that both the Klein-Gordon equation and the Dirac equation have solutions corresponding to states with negative energies. In Subsec. 1.4 we discuss how this issue can be addressed via the Feynman-Stueckelberg picture of causality. This picture reinterprets negative energy states by introducing the concept of antiparticle, and leads us to think of theories that incorporate quantum mechanics and relativity as theories for which particle number is not conserved.

1.2 The Klein-Gordon equation

We have seen in the previous subsection that the Klein-Gordon equation emerges from requiring the relativistic energy-momentum relation and taking the square of the hamiltonian operator in the position-space wave equation. In covariant form, the Klein-Gordon equation is given by

$$\left(\partial^2 + m^2\right)\phi(x) = 0 \ , \tag{1.15}$$

where

$$\partial^2 \equiv \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2. \tag{1.16}$$

The Klein-Gordon equation is relativistically covariant. That is, if we start with Eq. (1.15) and make a Lorentz transformation,

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu \ , \quad \Lambda^{\mu\rho}\Lambda^{\nu\sigma}g_{\mu\nu} = g^{\rho\sigma} \ , \tag{1.17}$$

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) \ , \tag{1.18}$$

in the primed coordinate system an equation of the same form holds, because

$$\begin{aligned}
(\partial^2 + m^2)\phi'(x) &= \left[(\Lambda^{-1})^\rho_\mu \partial_\rho (\Lambda^{-1})^\sigma_\nu \partial_\sigma g^{\mu\nu} + m^2\right]\phi(\Lambda^{-1}x) \\
&= (\partial_\rho \partial_\sigma g^{\rho\sigma} + m^2)\phi(\Lambda^{-1}x) = (\partial^2 + m^2)\phi(\Lambda^{-1}x) = 0 \ . \tag{1.19}
\end{aligned}$$

On the other hand, interpreting the Klein-Gordon equation as a single particle relativistic wave equation leads to difficulties with quantum mechanics. As the equation is second-order in the time derivative, the norm of ϕ is not conserved with time. We can see the difficulty by looking for a continuity equation for Klein-Gordon similar to Eq. (1.5) for the nonrelativistic case. Following the same steps as described for Eq. (1.5), we obtain

$$\partial_\mu j^\mu = 0 \ , \quad j^\mu = (\rho, \mathbf{j}) \ , \tag{1.20}$$

with

$$\rho = i\left[\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t}\phi\right] \ , \quad \mathbf{j} = -i\left[\phi^*(\boldsymbol{\nabla}\phi) - (\boldsymbol{\nabla}\phi^*)\phi\right] \ . \tag{1.21}$$

The current $\mathbf{j}$ is formally similar to that of the nonrelativistic case in Eq. (1.6). The density ρ, however, is not. Nor could it be, because $\phi^*\phi$ would transform under Lorentz like a scalar rather than like the time component of a four-vector. Because Eq. (1.15) contains second-order time derivatives, the density ρ contains terms in $\partial/\partial t$, and is not positive definite.

If we look for plane wave solutions of the Klein-Gordon equation,

$$\phi(x) = N e^{-ipx} \ , \tag{1.22}$$

where $p^\mu = (E, \mathbf{p})$, another difficulty arises. By substituting Eq. (1.22) into the equation, we find that Eq. (1.22) is solution if

$$p^2 = m^2 \ , \tag{1.23}$$

that is,

$$E = \pm\sqrt{\mathbf{p}^2 + m^2} \ . \tag{1.24}$$

The Klein-Gordon equation contains both positive-energy and negative-energy solutions. Since j^μ in Eq. (1.20) is proportional to p^μ, negative energy solutions also have negative probability density. We discuss in Subsec. 1.4 how to interpret negative-energy states.

1.3   The Dirac equation

We have seen in Eqs. (1.12),(1.13) that the Dirac equation has the form

$$i \, \frac{\partial}{\partial t} \, \psi = (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m) \, \psi \; . \tag{1.25}$$

Consistency with the relativistic energy-momentum relation requires that by squaring Eq. (1.25),

$$-\frac{\partial^2 \psi}{\partial t^2} = [-\alpha^i \alpha^j \nabla^i \nabla^j - i \, (\beta \alpha^i + \alpha^i \beta) m \nabla^i + \beta^2 m^2] \psi \; , \tag{1.26}$$

we reobtain the Klein-Gordon equation,

$$-\frac{\partial^2 \psi}{\partial t^2} = [-\nabla^i \nabla^i + m^2] \psi \; . \tag{1.27}$$

Then α^i and β must obey anticommutation relations,

$$\alpha^i \alpha^j + \alpha^j \alpha^i = 2\delta^{ij} \; , \quad \beta \alpha^i + \alpha^i \beta = 0 \; , \quad \beta^2 = 1 \; . \tag{1.28}$$

Eq. (1.28) implies that

$$\text{Tr } \alpha^i = \text{Tr } \beta = 0 \; , \tag{1.29}$$

and that the eigenvalues of α^i and β are ± 1. Then α^i and β must be even-dimensional matrices. In two dimensions there are no four matrices satisfying Eq. (1.28) (the three Pauli σ matrices would be three candidate matrices but there is no fourth anticommuting 2×2 matrix). So the minimum possible dimension is four. Then ψ in Eq. (1.25) is a four-component object, referred to as a four-component spinor.

A possible choice of α^i and β is given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.30}$$

where we use block matrix notation.

The Dirac equation can be recast in manifestly covariant form by defining four new matrices γ in terms of the $\boldsymbol{\alpha}$ and β as

$$\gamma^0 = \beta, \qquad \boldsymbol{\gamma} = \beta \boldsymbol{\alpha}, \tag{1.31}$$

and noting that Eq. (1.25), multiplied by β, can be compactly rewritten in terms of the γ matrices as

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \; , \tag{1.32}$$

where

$$\gamma^\mu = (\gamma^0, \boldsymbol{\gamma}) \; . \tag{1.33}$$

In terms of the γ matrices the anticommutation relations (1.28) become

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \; . \tag{1.34}$$

The representation (1.30) is

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.35}$$

Because the Dirac equation is first-order, it gives rise to a positive-definite density, unlike the Klein-Gordon equation. By manipulations analogous to those seen in the previous sections, we obtain the continuity equation

$$\partial_\mu j^\mu = 0 \ , \quad j^\mu = (\rho, \mathbf{j}) \ , \tag{1.36}$$

with

$$\rho = \psi^\dagger \psi, \qquad \mathbf{j} = \psi^\dagger \boldsymbol{\alpha} \psi. \tag{1.37}$$

On the other hand, because $\boldsymbol{\alpha}$ and β are traceless (Eq. (1.29)), the hamiltonian is traceless. Then the eigenvalues must be E, $-E$. Thus the Dirac equation, like the Klein-Gordon equation, has negative energy solutions.

A first interpretation of negative energy solutions is provided by Dirac's "sea" picture. Dirac postulates the existence of a "sea" of negative energy states (Fig. 1), such that the vacuum has all the negative energy states filled with electrons. The Pauli principle forbids any positive-energy electron from falling into one of the lower states. Although the vacuum state has infinite negative charge and energy, this leads to an acceptable theory based on the fact that all observations only involve differences in energy and charge. When energy is supplied and one of the negative energy electrons is promoted to a positive energy one, an electron-hole pair is created, i.e. a positive energy electron and a hole in the negative energy sea. The hole, i.e. the absence of a negative-energy electron, is seen as the presence of a positive-energy and positive-charge state, the positron.

This picture led Dirac to postulate (1927) the existence of the positron as the electron's antiparticle, which was discovered experimentally five years later.
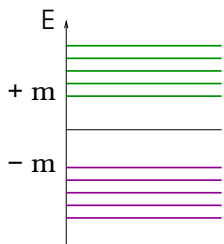

Figure 1: *Dirac sea picture of negative energy states.*

But Dirac's sea picture does not work for bosons, which have no exclusion principle. A second, more general interpretation of negative-energy states is given by Feynman's picture, which we describe in the next subsection.

1.4   The Feynman-Stueckelberg picture

The Feynman-Stueckelberg interpretation of negative-energy states does not appeal to the exclusion principle but rather to a causality principle. It is based on the observation that causality ensures that positive energy states, with time dependence e^{-iEt}, propagate forwards in time, and that if we impose that negative energy states propagate only backwards in time, with

$$e^{-i(-|E|)(-|t|)} \rightarrow e^{-i|E||t|} \ , \tag{1.38}$$

we still obtain an acceptable theory, consistent with causality. In this picture the emission of a negative energy particle with momentum p^μ is interpreted as the absorption of a positive energy antiparticle with momentum $-p^\mu$.

Consider for example photon-particle scattering (Fig. 2). In Fig. 2(a) a particle comes in with energy E_1, and at time t_1 and point x_1 it emits a photon with energy $E_\gamma < E_1$. It travels on forwards in time, and at time t_2 and point x_2 it absorbs the initial state photon, giving rise to the photon-particle final state.

Another process is shown in Fig. 2(b). In this process the particle coming in with energy E_1 emits a photon with energy $E_\gamma > E_1$, and is thus forced to travel backwards in time. Then at an earlier time it absorbs the initial state photon at the point x_2, which renders its energy positive again.
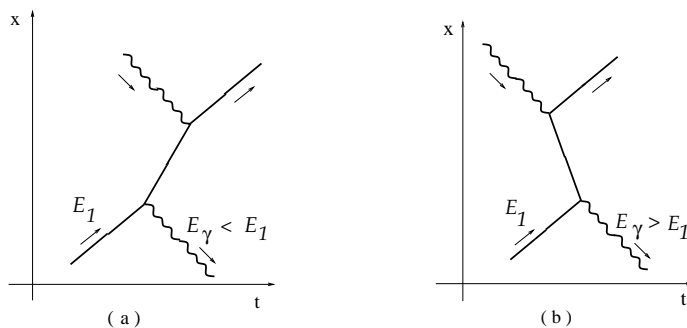

Figure 2: *Feynman interpretation of positive-energy and negative-energy states in terms of particle and antiparticle propagation.*

The process in Fig. 2(b) can be described by saying that in the initial state we have a particle and a photon, and that at point x_2 the photon creates a particle-antiparticle pair, both of which propagate forwards in time. The particle ends up in the final state, whereas the antiparticle is annihilated at a later time by the initial state particle, thereby producing the final state photon. According to this picture, the negative energy state moving backwards in time is viewed as a negatively charged state with positive energy moving forwards in time.

2 Spin

In the previous section we have introduced Dirac spinors, which we will use to describe spin-1/2 relativistic charges. In this section we further the study of spin in the context of Dirac theory.

We start from the algebraic description of the group of Lorentz transformations and associated representations, which parallels the treatment of the group of rotations and its representations in quantum mechanics. We describe how Dirac spinors emerge from this point of view, and we discuss solutions of the Dirac equations.

2.1 Algebra of Lorentz transformations

Let us start with the group of rotations in three dimensions. In quantum mechanics, given a particle with spin s, the matrices that rotate its n-component wave function, where $n = 2s + 1$, are constructed from the angular momentum operators J_i, $i = 1, 2, 3$. These satisfy the commutation relations

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \ . \tag{2.1}$$

Rotation operators are obtained by exponentiation as

$$R = e^{-i\theta_i J_i} \ , \tag{2.2}$$

where the parameters θ_i specify the rotation axis and angle. The angular momentum operators J_i are the generators of the rotation group, and have matrix representations for every dimensionality n. The representation for $n = 2$, corresponding to spin $s = 1/2$, is given by

$$J_i \rightarrow \frac{1}{2}\, \sigma_i \ , \tag{2.3}$$

where σ_i are the three Pauli matrices, so that

$$R_{1/2} = e^{-i\theta_i \sigma_i/2} \ . \tag{2.4}$$

We can generalize this from the group of rotations to the group of Lorentz transformations. One way to obtain the commutation relations of the Lorentz generators is as follows. In the case of the rotation group, the relations (2.1) can be obtained by taking

$$\boldsymbol{J} = \boldsymbol{x} \wedge \boldsymbol{p} = -i\boldsymbol{x} \wedge \boldsymbol{\nabla} \ , \tag{2.5}$$

and evaluating the commutators. The components of angular momentum in Eq. (2.5) can also be rewritten explicitly, using antisymmetric tensor notation, as

$$J^{ij} = -i(x^i \nabla^j - x^j \nabla^i) \ . \tag{2.6}$$

The generalization of this formula to the four-dimensional case,

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \ , \tag{2.7}$$

gives the correct generators of Lorentz transformations. As it is antisymmetric in μ and ν, Eq. (2.7) contains six operators, the three generators of rotations and three

generators of boosts. The commutation relations can be obtained by evaluating directly the commutators of the operators (2.7), and the result is

$$
\begin{aligned}
[L_{\mu\nu}, L_{\rho\sigma}] &= i\left(g_{\mu\sigma}L_{\nu\rho} - g_{\nu\sigma}L_{\mu\rho} - g_{\mu\rho}L_{\nu\sigma} + g_{\nu\rho}L_{\mu\sigma}\right) \\
&= i\left(g_{\mu\sigma}L_{\nu\rho} - \{\mu \leftrightarrow \nu\}\right) - \{\rho \leftrightarrow \sigma\} \ . \tag{2.8}
\end{aligned}
$$

Given the six operators in Eq. (2.7), it is helpful to distinguish the three operators

$$
J_i = \frac{1}{2}\varepsilon_{ijk}L_{jk} \tag{2.9}
$$

and the three operators

$$
K_i = L_{0i} \ . \tag{2.10}
$$

From Eq. (2.8) we get

$$
[J_i, J_j] = i\varepsilon_{ijk}J_k \ , \tag{2.11}
$$

that is, the three operators in Eq. (2.9) are the generators of rotations. The three remaining operators in Eq. (2.10) are the generators of boosts. From Eq. (2.8) we have

$$
[J_i, K_j] = i\varepsilon_{ijk}K_k \ , \tag{2.12}
$$

$$
[K_i, K_j] = -i\varepsilon_{ijk}J_k \ . \tag{2.13}
$$

The J and K operators in Eqs. (2.9),(2.10) have the clear physical interpretation of rotation and boost generators, but they are mixed by the algebra of commutation relations (2.8), as shown in Eqs. (2.12),(2.13). It is useful to disentangle the algebra by introducing the linear combinations of generators

$$
A_i = \frac{1}{2}(J_i + iK_i) \ , \quad B_i = \frac{1}{2}(J_i - iK_i) \ . \tag{2.14}
$$

By computing the commutators of the A and B operators in Eq. (2.14), we find

$$
[A_i, A_j] = i\varepsilon_{ijk}A_k \ , \qquad [B_i, B_j] = i\varepsilon_{ijk}B_k \ , \qquad [A_i, B_j] = 0 \ . \tag{2.15}
$$

That is, the A and the B operators do not mix, and each set obeys commutation relations of the form (2.1). This means that we can specify a representation of the group of Lorentz transformations by specifying a pair of rotation-group representations,

$$
(a, b) \ , \quad \text{where} \quad A_i A_i = a(a+1) \ , \quad B_i B_i = b(b+1) \ , \tag{2.16}
$$

with a and b integer or half-integer. Here $a + b$ gives the spin quantum number: so the representation $(0, 0)$ is spin-0 (scalar); $(1/2, 1/2)$ is spin-1 (vector); $(1/2, 0)$ and $(0, 1/2)$ are spin-1/2. The latter are referred to as Weyl spinors (respectively, left-handed and right-handed). A Dirac spinor is obtained from two Weyl spinors, $(1/2, 0) \oplus (0, 1/2)$.

Dirac spinors are thus identified by a representation of the group of Lorentz transformations reducible to the sum of two representations (2.16), $(1/2, 0)$ and $(0, 1/2)$.

We discuss Weyl and Dirac spinors in the next two subsections.

2.2   Weyl spinors

A left-handed Weyl spinor is defined by taking $a = 1/2, b = 0$ in Eq. (2.16). We thus have

$$A^i = \frac{1}{2}\,\sigma^i \ , \qquad B^i = 0 \ . \tag{2.17}$$

Using Eq. (2.14), the representation of the rotation and boost generators is given by

$$J^i = \frac{1}{2}\,\sigma^i \ \text{(rotation generators)} \ , \qquad K^i = -i\,\frac{1}{2}\,\sigma^i \ \text{(boost generators)} . \tag{2.18}$$

So a left-handed Weyl spinor is a two-component spinor,

$$\xi_L = \begin{pmatrix} \xi_L^1 \\ \xi_L^2 \end{pmatrix} \ , \tag{2.19}$$

which transforms under rotations and boosts as

$$\xi_L \rightarrow e^{-i\theta^k \sigma^k/2 - \eta^k \sigma^k/2}\xi_L \ . \tag{2.20}$$

A right-handed Weyl spinor is defined by taking $a = 0, b = 1/2$ in Eq. (2.16). Then

$$A^i = 0 \ , \qquad B^i = \frac{1}{2}\,\sigma^i \ , \tag{2.21}$$

i.e.,

$$J^i = \frac{1}{2}\,\sigma^i \ \text{(rotation generators)} \ , \qquad K^i = i\,\frac{1}{2}\,\sigma^i \ \text{(boost generators)} . \tag{2.22}$$

So a right-handed Weyl spinor is a two-component spinor,

$$\xi_R = \begin{pmatrix} \xi_R^1 \\ \xi_R^2 \end{pmatrix} \ , \tag{2.23}$$

which transforms under rotations and boosts as

$$\xi_R \rightarrow e^{-i\theta^k \sigma^k/2 + \eta^k \sigma^k/2}\xi_R \ . \tag{2.24}$$

Eq. (2.20) and Eq. (2.24) differ by the sign in the boost transformation.

2.3   Dirac spinors

A Dirac spinor is a four-component spinor built out of two Weyl spinors as

$$\psi = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} \ . \tag{2.25}$$

We can construct explicitly its Lorentz transformation matrix S

$$\psi \rightarrow \psi' = S\psi \tag{2.26}$$

from those for the Weyl spinors in Sec. 2.2. We obtain

$$\psi \to \psi' = e^{-i\omega_{\mu\nu}\Sigma^{\mu\nu}}\psi \equiv S\psi \ , \tag{2.27}$$

where

$$\Sigma^{\mu\nu} \equiv \frac{i}{4}\, [\gamma^\mu, \gamma^\nu] \tag{2.28}$$

with the γ matrix representation

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.29}$$

Thus the generators of boosts for Dirac spinors are

$$\Sigma^{0i} = \frac{i}{4}\, [\gamma^0, \gamma^i] = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \quad (\text{boost generators}) \ , \tag{2.30}$$

and the generators of rotations are

$$\Sigma^{ij} = \frac{i}{4}\, [\gamma^i, \gamma^j] = \frac{1}{2}\, \varepsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad (\text{rotation generators}) \ . \tag{2.31}$$

An alternative method, equivalent to that given above, for constructing Dirac spinors is based on observing that if 4 $n \times n$ matrices γ^μ satisfy

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \ , \tag{2.32}$$

then

$$\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{2.33}$$

obey the Lorentz algebra (2.8). Then, by a reasoning similar to that followed in Sec. 1.3, one sees that n must be 4. By this method one arrives at Dirac spinors without going through the construction from two Weyl spinors.

Note the following two properties of the Lorentz transformation matrix S for Dirac spinors, which follow from Eqs. (2.27),(2.28). The first is that

$$S^{-1}\gamma^\mu S = \Lambda^{\mu\nu}\gamma_\nu \ . \tag{2.34}$$

This relation implies that the Dirac equation Eq. (1.32),

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \ , \tag{2.35}$$

is relativistically covariant, because under a Lorentz transformation we have

$$\begin{aligned}
(i\gamma^\mu\partial_\mu - m)\psi(x) \quad &\to \quad \left[i\gamma^\mu(\Lambda^{-1})^\rho_\mu\partial_\rho - m\right] S\psi(\Lambda^{-1}x) \\
&= \quad S(i\gamma^\nu\partial_\nu - m)\psi(\Lambda^{-1}x) = 0 \ .
\end{aligned} \tag{2.36}$$

The second property is that, because boost generators (2.30) are not hermitian, S is not unitary; rather, it satisfies

$$S^\dagger = \gamma^0 S^{-1}\gamma^0 \ . \tag{2.37}$$

For this reason the product $\psi^\dagger\psi$ is not Lorentz invariant. It is thus useful to define the adjoint spinor

$$\overline{\psi} \equiv \psi^\dagger\gamma^0 \ . \tag{2.38}$$

Using Eq. (2.37), we see that the product $\overline{\psi}\psi$ is Lorentz invariant.

We have seen the transformation law of Dirac spinors under rotations and boosts. Lorentz transformations also include discrete transformations, space parity and time reversal. The transformation law of Dirac spinors under these are

$$P\psi(\boldsymbol{x},t)P^{-1} = \eta\gamma^0\psi(-\boldsymbol{x},t) \ , \qquad P\overline{\psi}(\boldsymbol{x},t)P^{-1} = \eta^*\overline{\psi}(-\boldsymbol{x},t)\gamma^0 \ , \tag{2.39}$$

where η is a phase factor to be fixed, and

$$T\psi(\boldsymbol{x},t)T^{-1} = -\gamma^1\gamma^3\psi(\boldsymbol{x},-t) \ , \qquad T\overline{\psi}(\boldsymbol{x},t)T^{-1} = \overline{\psi}(\boldsymbol{x},-t)\gamma^1\gamma^3 \ . \tag{2.40}$$

A third discrete symmetry is charge conjugation, exchanging particle and antiparticle,

$$C\psi C^{-1} = -i\gamma^2\psi^* \ , \qquad C\overline{\psi}C^{-1} = -i\psi^T\gamma^2\gamma^0 \ . \tag{2.41}$$

Finally, it is useful to introduce a fifth γ matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \ , \tag{2.42}$$

obeying

$$\left(\gamma^5\right)^2 = 1 \ , \qquad \{\gamma^5,\gamma^\mu\} = 0 \ , \qquad \left(\gamma^5\right)^\dagger = \gamma^5 \ . \tag{2.43}$$

Then define the projection operators

$$P_L = \frac{1-\gamma^5}{2} \ , \qquad P_R = \frac{1+\gamma^5}{2} \ . \tag{2.44}$$

In the representation (2.29) we have

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \implies P_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \qquad P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{2.45}$$

Thus

$$\psi_L \equiv P_L\psi = \frac{1-\gamma^5}{2}\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} = \begin{pmatrix} \xi_L \\ 0 \end{pmatrix} \ , \quad \psi_R \equiv P_R\psi = \frac{1+\gamma^5}{2}\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \xi_R \end{pmatrix} \ . \tag{2.46}$$

Note that, because γ^5 anticommutes with γ^0, for the adjoint we have

$$\overline{\psi}_L = \psi_L^\dagger\gamma^0 = \psi^\dagger P_L\gamma^0 = \psi^\dagger\gamma^0 P_R = \overline{\psi}P_R \ , \tag{2.47}$$

and similarly

$$\overline{\psi}_R = \overline{\psi}P_L \ . \tag{2.48}$$

By including γ^5, it is possible to see that any combination of the form

$$\overline{\psi}\Gamma\psi \ , \tag{2.49}$$

where Γ is any 4×4 matrix, can be decomposed into terms with definite transformation properties under Lorentz, because a basis for 4×4 matrices is given by the sixteen, linearly independent matrices

$$\mathbf{1} \ , \quad \gamma^5 \ , \quad \gamma^\mu \ , \quad \gamma^\mu\gamma^5 \ , \quad \Sigma^{\mu\nu} \ , \tag{2.50}$$

transforming respectively like scalar, pseudoscalar, vector, pseudovector and tensor.

2.4 Solutions of the Dirac equation

We can use Dirac spinors to write plane wave solutions of the Dirac equation. Consider

$$\psi(x) = \begin{pmatrix} \chi(\mathbf{p}) \\ \phi(\mathbf{p}) \end{pmatrix} e^{-ipx} \ , \tag{2.51}$$

where $p^\mu = (E, \mathbf{p})$, and χ and ϕ are two-components spinors. Substituting (2.51) into the Dirac equation (2.35) and using the representation (1.35) yields the coupled equations for χ and ϕ

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} \ , \tag{2.52}$$

that is,

$$\boldsymbol{\sigma} \cdot \mathbf{p}\, \phi = (E - m)\, \chi$$
$$\boldsymbol{\sigma} \cdot \mathbf{p}\, \chi = (E + m)\, \phi \ . \tag{2.53}$$

These have solutions for positive and negative energies, $E = \pm\sqrt{\mathbf{p}^2 + m^2}$. We can write the solution for positive energies as

$$\psi_+(x) = \mathcal{N} \begin{pmatrix} \chi_r \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \chi_r \end{pmatrix} e^{-ipx} \equiv u_r(\mathbf{p}) e^{-ipx} \ , \tag{2.54}$$

where $\mathcal{N} = \sqrt{E + m}$, and the spinors χ_r for $r = 1, 2$ are given by

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{2.55}$$

For negative energies, it is convenient to make the transformation $p^\mu \to -p^\mu$, so that we write the corresponding solution as

$$\psi_-(x) = \mathcal{N} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \chi_r \\ \chi_r \end{pmatrix} e^{ipx} \equiv v_r(\mathbf{p}) e^{ipx} \ . \tag{2.56}$$

The spinors u and v defined by Eqs. (2.54),(2.56) correspond respectively to particle and antiparticle solutions. Taking the solutions in the rest frame $\mathbf{p} = 0$, the top two components of ψ describe electrons with spin up and spin down, while the bottom two components describe positrons with spin up and spin down. This provides a clear physical interpretation to the four components of Dirac spinors. For arbitrary $\mathbf{p}$, we can study the spin content of the solutions by using the explicit expression of the spin operator

$$\boldsymbol{S} = \frac{1}{2}\, \boldsymbol{\Sigma} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \ , \tag{2.57}$$

corresponding to spin $1/2$,

$$\boldsymbol{S}^2 = \frac{1}{4}\, \boldsymbol{\Sigma}^2 = \frac{1}{4} \begin{pmatrix} \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \end{pmatrix} = \frac{3}{4}\, \mathbf{1} \ , \tag{2.58}$$

and helicity operator

$$h = \frac{\boldsymbol{S} \cdot \boldsymbol{p}}{|\boldsymbol{p}|} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}} \end{pmatrix} \ . \tag{2.59}$$

The u and v spinors satisfy the Dirac equation in momentum space,

$$(\not{p} - m)u = 0 \ , \quad (\not{p} + m)v = 0 \ , \tag{2.60}$$

and obey orthonormality and completeness relations. The orthonormality relations are given by

$$u_r^\dagger(\mathbf{p})u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p})v_s(\mathbf{p}) = 2E\delta^{rs} \ , \tag{2.61}$$

$$u_r^\dagger(\mathbf{p})v_s(-\mathbf{p}) = v_r^\dagger(\mathbf{p})u_s(-\mathbf{p}) = 0 \ . \tag{2.62}$$

Equivalently, in terms of the adjoint spinors $\bar{u} = u^\dagger\gamma^0$ and $\bar{v} = v^\dagger\gamma^0$,

$$\bar{u}_r(\mathbf{p})u_s(\mathbf{p}) = -\bar{v}_r(\mathbf{p})v_s(\mathbf{p}) = 2m\delta^{rs} \ , \tag{2.63}$$

$$\bar{u}_r(\mathbf{p})v_s(\mathbf{p}) = -\bar{v}_r(\mathbf{p})u_s(\mathbf{p}) = 0 \ , \tag{2.64}$$

The completeness relations are given by

$$\sum_{r=1}^{2} u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) = (\not{p} + m) \ , \tag{2.65}$$

$$\sum_{r=1}^{2} v_r(\mathbf{p})\bar{v}_r(\mathbf{p}) = (\not{p} - m) \ . \tag{2.66}$$

3 Perturbation theory and S matrix

We now consider the coupling of the Klein-Gordon and the Dirac equation to electromagnetism, and study scattering processes (Fig. 3) in quantum electrodynamics, applying time-dependent perturbation theory.

We will express physical cross sections in terms of invariant scattering matrix elements. The application of perturbation theory can be encoded in the Feynman rules for the calculation of the S matrix elements.
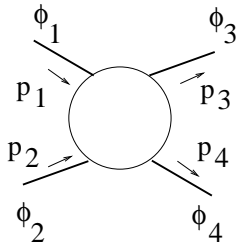
Figure 3: *The scattering process $\phi_1 + \phi_2 \to \phi_3 + \phi_4$.*

3.1 Electromagnetic interaction of spinless charges

We consider a relativistic, spinless system described by the Klein-Gordon equation, Eq. (1.15), and we couple it to electromagnetism via the replacement

$$\partial_\mu \to \partial_\mu + ieA_\mu \ . \tag{3.1}$$

By including the electromagnetic interaction (3.1) into Eq. (1.15), the equation of motion of the system can be written

$$(\partial^\mu \partial_\mu + m^2)\phi = -ie(\partial_\mu A^\mu + A_\mu \partial^\mu)\phi + e^2 A^2 \phi \equiv -\mathcal{V}\phi \ , \tag{3.2}$$

where in the right hand side we identify the potential

$$\mathcal{V} = V_1 + V_2 \ , \tag{3.3}$$

$$V_1 = ie(\partial_\mu A^\mu + A^\mu \partial_\mu) \ , \tag{3.4}$$

$$V_2 = -e^2 A^2 \ . \tag{3.5}$$

Let us apply first-order time-dependent perturbation theory to the transition amplitude $\mathcal{A}$ due to the interaction given by $\mathcal{V}$,

$$\mathcal{A} = -i \int d^3x \ dt \ \phi_f^* \mathcal{V}\phi_i \ . \tag{3.6}$$

We will derive the result for the term V_1 of the potential. The contribution of V_2 can be treated analogously, and we will include the result later.

By inserting the potential (3.4) into Eq. (3.6) and doing an integration by parts, we can recast the transition amplitude in the form of a $j \cdot A$ interaction,

$$\mathcal{A} = -i \int d^4x \; j_\mu A^\mu \; , \tag{3.7}$$

where the interaction current is given by

$$j_\mu = ie[\phi_3^* [\partial_\mu \phi_1) - (\partial_\mu \phi_3^*)\phi_1] \; . \tag{3.8}$$

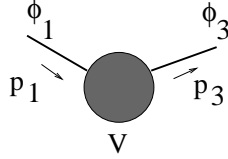Figure 4: *The $\phi_1 \to \phi_3$ transition subprocess by $\mathcal{V}$ interaction.*

We take the initial and final states ϕ_1 and ϕ_3 to be given by plane waves (Fig. 4)

$$\phi_1 = N_1 e^{-ip_1 x} \; , \qquad \phi_3 = N_3 e^{-ip_3 x} \; , \tag{3.9}$$

normalized in a box of volume V so that

$$N_1 = 1/\sqrt{2E_1 V}, \quad N_3 = 1/\sqrt{2E_3 V} \; . \tag{3.10}$$

Inserting Eqs. (3.4),(3.9) into Eq. (3.6) we obtain

$$\begin{aligned} \mathcal{A} &= -iN_1 N_3 \int d^4x \; e^{ip_3 x}(ie)(\partial_\mu A^\mu + A^\mu \partial_\mu)e^{-ip_1 x} \\ &= -ieN_1 N_3 (p_1 + p_3)_\mu \int d^4x \; e^{-iqx} A^\mu \; , \end{aligned} \tag{3.11}$$

where $q = p_1 - p_3$, and we have done an integration by parts in the second line.

Let us now determine the electromagnetic potential A^μ in Eq. (3.11) which results from the transition $\phi_2 \to \phi_4$. Let us use the Lorentz gauge-fixing condition,

$$\partial^\nu A_\nu = 0 \; . \tag{3.12}$$

Then the equation of motion for the electromagnetic potential is given by

$$\partial^2 A^\mu = J^\mu \; , \tag{3.13}$$

where the current J^μ is of the form (3.8), with the replacements $1 \to 2$, $3 \to 4$. The states ϕ_2 and ϕ_4 are also represented by plane waves, analogously to Eqs. (3.9),(3.10). By Fourier-transforming Eq. (3.13) with respect to x, we get

$$-q^2 \widetilde{A}^\mu = \widetilde{J}^\mu \; , \tag{3.14}$$

where

$$\widetilde{A}^\mu = \int d^4x \; e^{-iqx} A^\mu \; , \quad \widetilde{J}^\mu = \int d^4x \; e^{-iqx} J^\mu \; . \tag{3.15}$$

That is,

$$\tilde{A}^\mu = \frac{-g^{\mu\nu}}{q^2} \; \tilde{J}_\nu \; . \qquad (3.16)$$

By substituting the explicit expression of the current J and inserting the result (3.16) into Eq. (3.11), we obtain

$$\mathcal{A} = ie^2 N_1 N_3 N_2 N_4 (p_1 + p_3)_\mu (p_2 + p_4)^\mu \; \frac{1}{q^2} \int d^4x \; e^{i(p_3 - p_1)x} e^{i(p_4 - p_2)x} \; . \qquad (3.17)$$
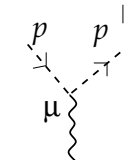
Performing the integral in d^4x in Eq. (3.17) gives $(2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2)$, which expresses four-momentum conservation in the scattering process (Fig. 3). By inserting this result and the explicit expressions of the normalization factors, we can rewrite Eq. (3.17) as

$$\mathcal{A} = (2\pi)^4 \delta^4(P_f - P_i) \; \frac{1}{\prod_f \sqrt{2E_f V}} \; \frac{1}{\prod_i \sqrt{2E_i V}} \; \mathcal{M}_{fi} \; , \qquad (3.18)$$

where the products over i and f run respectively over initial and final state particles, P_i and P_f denote the total four-momentum in the initial and final state, and we have defined the scattering matrix element

$$\mathcal{M}_{fi} = -e \; (p_1 + p_3)_\mu \; \frac{-i \; g^{\mu\nu}}{q^2} \; e \; (p_2 + p_4)_\nu \; . \qquad (3.19)$$

We will interpret Eq. (3.19) as resulting from associating a factor $ie(p_1 + p_3)_\mu$ to the

Figure 5: *Feynman rules for interaction vertices of spin-0 particles with photons.*

transition vertex in Fig. 4, an analogous factor $ie(p_2 + p_4)_\mu$ to the $\phi_2 \to \phi_4$ transition vertex, and the factor $-ig^{\mu\nu}/q^2$ to the electromagnetic interaction. This gives the Feynman rules in the top row in Fig. 5 and in the top row in Fig. 8 (more comment on this in Sec. 3.3). An analogous treatment of the potential term in Eq. (3.5) gives the Feynman rule in the bottom row in Fig. 5.

3.2 Electromagnetic interaction of spin-1/2 charges

The case of spin 1/2 can be treated by an analysis analogous to that of the previous subsection. In this case, by including the electromagnetic interaction (3.1) into the Dirac equation (1.32),

$$(i\slashed{\partial} - e\slashed{A} - m)\psi = 0 \; , \qquad (3.20)$$

we arrive at the interaction potential

$$\mathcal{V} = -e\gamma^0\gamma^\mu A_\mu \ . \tag{3.21}$$

Then first-order perturbation theory gives

$$
\begin{aligned}
\mathcal{A} &= -i \int d^3x \ dt \ \psi_f^* \ \mathcal{V} \ \psi_i \\
&= ie \int d^4x \ \overline{\psi}_f \gamma^\mu \psi_i A_\mu \ ,
\end{aligned}
\tag{3.22}
$$

that is, a $j \cdot A$ interaction,

$$\mathcal{A} = -i \int d^4x \ j^\mu A_\mu \ , \tag{3.23}$$

with the current given by

$$j^\mu = -e\overline{\psi}_f\gamma^\mu\psi_i \ . \tag{3.24}$$

Now we represent initial and final states by plane-wave solutions

$$\psi_k = N_k u(p_k)e^{-ip_k x} \ , \quad N_k = 1/\sqrt{2E_k V}, \quad k = 1,2,3,4 \ . \tag{3.25}$$

By following the same steps as in the scalar case, we arrive at

$$\mathcal{A} = (2\pi)^4\delta^4(P_f - P_i) \prod_f \left[\frac{1}{\sqrt{2E_f V}} \right] \prod_i \left[\frac{1}{\sqrt{2E_i V}} \right] \mathcal{M}_{fi} \ , \tag{3.26}$$

where now the scattering matrix element is given by

$$\mathcal{M}_{fi} = -e\overline{u}(p_3)\gamma_\mu u(p_1) \frac{-i \ g^{\mu\nu}}{q^2} \ e\overline{u}(p_4)\gamma_\nu u(p_2) \ . \tag{3.27}$$

The corresponding Feynman rule for the spin-1/2 transition vertex is given in Fig. 5.

Figure 6: *Feynman rule for the interaction vertex of spin-1/2 particles with photons.*

3.3 Green's functions

In Sec. 3.1 we have treated the equation of motion (3.13) for the electromagnetic potential by taking the Fourier transformation, and we have solved for the potential in Eq. (3.16) in terms of the Green function, or propagator, of the ∂^2 differential operator, proportional to $1/q^2$. This function has poles for $q^2 = q^{02} - \mathbf{q}^2 = 0$, that is,

$$q^0 = \pm|\mathbf{q}| \ . \tag{3.28}$$

Therefore, when we take the inverse Fourier transform, in order to fully specify the solution we need to specify the prescription for going around these poles on the integration contour in the complex q^0 plane. Different possible choices are sketched in Fig. 7, and correspond to different boundary conditions on the solutions of Eq. (3.13):

a) vanishing fields far in the past (radiation case)

b) vanishing fields far in the future (absorption case)

c) propagation of positive frequencies in the future and negative frequencies in the past (Feynman)

d) propagation of negative frequencies in the future and positive frequencies in the past (anti-Feynman)
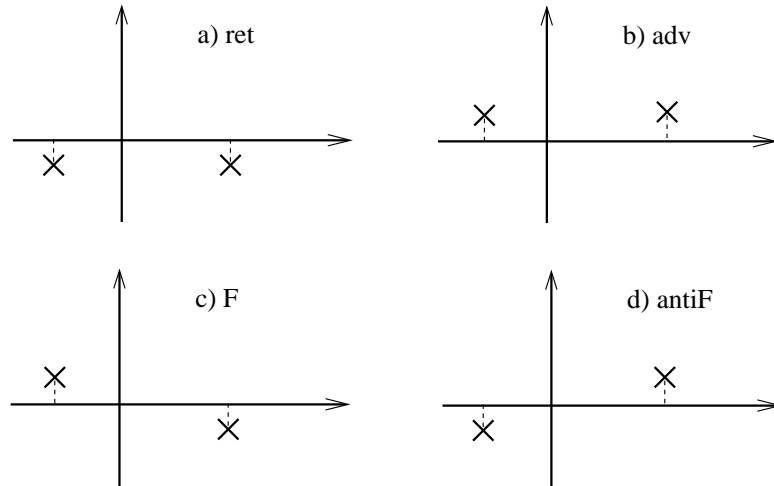
Figure 7: *Contours in the complex q^0 plane: a) retarded; b) advanced; c) Feynman; d) anti-Feynman.*

The Feynman contour is obtained by taking

$$\frac{1}{q^2 + i\varepsilon} \tag{3.29}$$

in the denominator of the Green function, where ε is real and positive. We will take the Feynman prescription for the propagator, according to the causality picture in Sec. 1.4.

The same discussion applies to the Green functions for the the Klein-Gordon equation and the Dirac equation. The results for the photon, Klein-Gordon and Dirac propagators are given in Fig. 8, using the Feynman prescription.

Note that Fig. 8 gives the photon propagator for a general covariant gauge-fixing condition, Eq. (3.12). This depends on the gauge parameter ξ. The value $\xi = 1$ is the Feynman gauge choice.

$$\mu \,\wwww\, \nu \qquad -\,i\;\frac{g^{\mu\nu} - (1-\xi)\,q^{\mu}q^{\nu}/q^{2}}{q^{2}+i\,\varepsilon}$$

$$\frac{i}{q^{2}-m^{2}+i\,\varepsilon}$$

$$\frac{i\,(\slashed{q}+m)}{q^{2}-m^{2}+i\,\varepsilon}$$

Figure 8: *Feynman rules for propagators: (top) photon; (middle) Klein-Gordon; (bottom) Dirac.*

3.4   From scattering matrix elements to cross sections

In order to go from transition amplitudes to scattering cross sections, we need to i) construct transition probabilities by squaring the amplitudes, ii) integrate over the final state phase space, iii) divide by the incident flux of particles.

To carry out step i), we evaluate the square of the δ function, working in volume V and time interval T, as

$$
\begin{aligned}
|(2\pi)^{4}\delta^{4}(P_{f}-P_{i})|^{2} &\simeq (2\pi)^{4}\delta^{4}(P_{f}-P_{i})\int e^{i(P_{f}-P_{i})x}\,d^{4}x\\
&\simeq VT\,(2\pi)^{4}\delta^{4}(P_{f}-P_{i})\ .
\end{aligned}
\tag{3.30}
$$

For step ii), we take the phase space element for each particle f in the final state,

$$
d\phi_{f}=\frac{V d^{3}p_{f}}{(2\pi)^{3}}\ .
\tag{3.31}
$$

Then the transition probability per unit time is given by

$$
\begin{aligned}
dw_{fi} &= \frac{|\mathcal{A}_{fi}|^{2}}{T}\prod_{f}d\phi_{f}\\
&= \frac{1}{T}\prod_{f}\left[\frac{1}{2E_{f}V}\right]\prod_{i}\left[\frac{1}{2E_{i}V}\right](2\pi)^{4}\delta^{4}(P_{f}-P_{i})|\mathcal{M}_{fi}|^{2}VT\prod_{f}\left(\frac{V d^{3}p_{f}}{(2\pi)^{3}}\right).
\end{aligned}
\tag{3.32}
$$

Note that the factor of T cancels; so does each factor of V associated with final state particles.

Let us now consider the decay process in Fig. 9. The decay rate is given by

$$
d\Gamma_{fi} = \frac{1}{2E_{i}V}|\mathcal{M}_{fi}|^{2}V\prod_{f}\left(\frac{d^{3}p_{f}}{(2\pi)^{3}2E_{f}}\right)(2\pi)^{4}\delta^{4}(P_{f}-P_{i})
$$

$$= \quad \frac{1}{2E_i} |\mathcal{M}_{fi}|^2 \prod_f \left(\frac{d^3 p_f}{(2\pi)^3 2E_f} \right) (2\pi)^4 \delta^4 (P_f - P_i) \ . \tag{3.33}$$

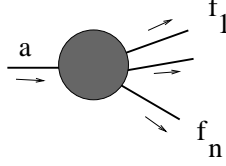Note the cancellation of the factor of V for the initial state.

Figure 9: *Decay process $a \to f_1 + f_2 + \ldots f_n$.*

Next consider the scattering process in Fig. 10. To obtain the cross section we need to divide the transition probability by the incident particle flux F,

$$d\sigma_{fi} = \frac{dw_{fi}}{F} \ , \tag{3.34}$$

$$F = \frac{1}{V} |v_a - v_b| \ , \tag{3.35}$$

where $|v_a - v_b|$ is the relative velocity of colliding particles. Thus

$$d\sigma_{fi} \quad = \quad \frac{V}{|v_a - v_b|} \frac{1}{2E_a V 2E_b V} |\mathcal{M}_{fi}|^2 V \prod_f \left(\frac{d^3 p_f}{(2\pi)^3 2E_f} \right) (2\pi)^4 \delta^4 (P_f - P_i)$$

$$= \quad \frac{1}{4E_a E_b |v_a - v_b|} |\mathcal{M}_{fi}|^2 \prod_f \left(\frac{d^3 p_f}{(2\pi)^3 2E_f} \right) (2\pi)^4 \delta^4 (P_f - P_i) \ . \tag{3.36}$$

Note again the cancellation of all V factors.
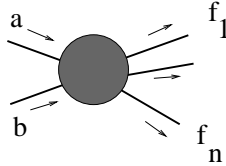
Figure 10: *Scattering process $a + b \to f_1 + f_2 + \ldots f_n$.*

Therefore the scattering cross section has the form

$$d\sigma = \frac{1}{\mathcal{J}} |\mathcal{M}_{fi}|^2 \, d\Phi \ , \tag{3.37}$$

22

where each of the three factors is relativistically invariant. The invariant matrix element square $|\mathcal{M}_{fi}|^2$ contains the dynamics of the process, while the factors $\mathcal{J}$ and $d\Phi$ are kinematic factors, giving respectively the invariant initial-state flux,

$$\mathcal{J} = 4E_a E_b |v_a - v_b| \ , \tag{3.38}$$

and the invariant final-state phase space

$$d\Phi = \prod_f \left(\frac{d^3 p_f}{(2\pi)^3 2E_f} \right) (2\pi)^4 \delta^4 (P_f - P_i) \ . \tag{3.39}$$

The invariant flux $\mathcal{J}$ can also be rewritten in equivalent forms as

$$
\begin{aligned}
\mathcal{J} &= 4E_a E_b |v_a - v_b| \\
&= 4|p_i^{(c.m.)}|\sqrt{s} \\
&= 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} \ ,
\end{aligned}
\tag{3.40}
$$

where $s = (p_a + p_b)^2$, and $p_i^{(c.m.)}$ is the initial three-momentum in the center of mass frame.

4 Coulomb scattering

In this section we analyze Coulomb scattering in QED using the formalism developed in Sec. 3 for S-matrix calculations in perturbation theory. We begin in Subsec. 4.1 by recalling Coulomb scattering in the nonrelativistic case. In Subsecs. 4.2-4.4 we calculate elastic electron-muon scattering to lowest order in the QED coupling. Then in Subsec. 4.5 we obtain the cross section for the scattering of a relativistic particle from an external Coulomb potential. In Subsec. 4.6 we consider the annihilation of electron pairs into muon pairs, related to $e\mu$ scattering by crossing symmetry.

4.1 Nonrelativistic case

The cross section in nonrelativistic quantum mechanics for the scattering of a particle of mass m from potential V is given in the first-order Born approximation by

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2} \, |\widetilde{V}(\mathbf{q})|^2 \ , \tag{4.1}$$

where $\widetilde{V}$ is the Fourier transform of the potential,

$$\widetilde{V}(\mathbf{q}) = \int d^3x \ e^{-i\mathbf{q}\cdot\mathbf{x}} \ V(\mathbf{x}) \ , \tag{4.2}$$

$\boldsymbol{q}$ is the momentum transferred in the scattering (Fig. 11), and $d\Omega$ is the solid angle element.
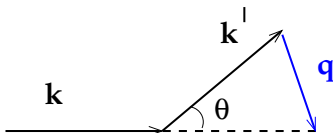

Figure 11: *Scattering through angle θ, with* $\mathbf{q} = \mathbf{k} - \mathbf{k}'$.

Let us take a potential of the form

$$V(\mathbf{x}) = C \, \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|} \ , \qquad \mu > 0 \ . \tag{4.3}$$

We can apply Eq. (4.1), and we can obtain the nonrelativistic cross section for Coulomb scattering by letting

$$C \to e^2 \ , \qquad \mu \to 0 \tag{4.4}$$

in the result.

By inserting Eq. (4.3) into Eq. (4.2), we get

$$\widetilde{V}(\mathbf{q}) = C \, \frac{4\pi}{\mu^2 + \mathbf{q}^2} \ . \tag{4.5}$$

Thus

$$\frac{d\sigma}{d\Omega} = \frac{4C^2 m^2}{(\mu^2 + \mathbf{q}^2)^2} \ . \tag{4.6}$$

Substituting Eq. (4.4) into Eq. (4.6), we obtain that the Coulomb scattering cross section in the nonrelativistic case is given by

$$
\left(\frac{d\sigma}{d\Omega}\right)_{\text{Coul.}} = \frac{4m^2 e^4}{\mathbf{q}^4}
$$

$$
= \frac{\alpha^2}{4\,\mathbf{k}^2\,\mathbf{v}^2\,\sin^4(\theta/2)} \equiv \left(\frac{d\sigma}{d\Omega}\right)_R \quad, \tag{4.7}
$$

where in the last line we have used $\mathbf{q}^2 = 4\mathbf{k}^2 \sin^2(\theta/2)$, $\mathbf{k} = m\mathbf{v}$. The result is given by the classical Rutherford scattering cross section $(d\sigma/d\Omega)_R$.

In the next few sections we analyze elastic electron-muon scattering in the fully relativistic quantum theory. From this analysis we will also obtain, in Subsec. 4.5, the relativistic correction to the result (4.7) for scattering from an external Coulomb potential.

4.2   The $e\mu$ scattering matrix element in QED

Consider elastic electron-muon scattering $e(p) + \mu(k) \to e(p') + \mu(k')$ (Fig. 12). Let m be the electron mass and M the muon mass.

Using the Feynman rules for perturbation theory in Sec. 3, the scattering matrix element $\mathcal{M}_{fi}$ is given by

$$
\mathcal{M}_{fi} = ie^2 \overline{u}_{r'}(k')\gamma^\mu u_r(k)\frac{g_{\mu\nu}}{q^2}\overline{u}_{s'}(p')\gamma^\nu u_s(p) \quad, \tag{4.8}
$$

where the momentum and spin labels are given in Fig. 12, and $q^2 = (k - k')^2$ is the invariant momentum transfer.
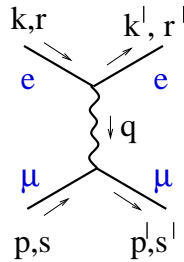

Figure 12: *Electron-muon scattering at lowest order in e.*

To compute the unpolarized cross section, we need the squared matrix element, averaged over initial spins and summed over final spins:

$$
\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{2}\sum_{r=1}^{2}\frac{1}{2}\sum_{s=1}^{2}\sum_{r'=1}^{2}\sum_{s'=1}^{2}|\mathcal{M}_{fi}|^2
$$

$$
= \frac{1}{4}\frac{e^4}{(q^2)^2}\sum_{spins}|\overline{u}_{r'}(k')\gamma^\mu u_r(k)|^2|\overline{u}_{s'}(p')\gamma_\mu u_s(p)|^2 \quad. \tag{4.9}
$$

In the next subsection we give the basic result that is needed to evaluate such spin sums.

4.3 Fermionic spin sums

For any matrix Γ given by a product of Dirac γ matrices, it can be shown that

$$\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} |\overline{u}_\alpha(p')\Gamma u_\beta(p)|^2 = \text{Tr}\left[\Gamma(\not{p}+m)\gamma^0\Gamma^\dagger\gamma^0(\not{p}'+m)\right] \quad . \tag{4.10}$$

To see this, write the square as

$$\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} |\overline{u}_\alpha(p')\Gamma u_\beta(p)|^2 = \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} [\overline{u}_\alpha(p')\Gamma u_\beta(p)][\overline{u}_\alpha(p')\Gamma u_\beta(p)]^* \tag{4.11}$$

and evaluate the complex conjugate factor:

$$\begin{aligned}
[\overline{u}_\alpha(p')\Gamma u_\beta(p)]^* &= u_\beta^\dagger(p)\Gamma^\dagger(u_\alpha^\dagger(p')\gamma^0)^\dagger \\
&= \overline{u}_\beta(p)\underbrace{\gamma^0\Gamma^\dagger\gamma^0}_{\widetilde{\Gamma}}u_\alpha(p') \quad .
\end{aligned} \tag{4.12}$$

Now write out all products of spinors and γ matrices in Eq. (4.11) in components:

$$\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} |\overline{u}_\alpha(p')\Gamma u_\beta(p)|^2 = \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} \overline{u}_{\alpha a}(p')\Gamma_{ab}u_{\beta b}(p)\overline{u}_{\beta c}(p)\widetilde{\Gamma}_{cd}u_{\alpha d}(p') \tag{4.13}$$

Next use the completeness relation for u spinors:

$$\sum_{\alpha=1}^{2} u_{\alpha j}(p)\overline{u}_{\alpha k}(p) = (\not{p}+m)_{jk} \quad . \tag{4.14}$$

Then from Eq. (4.13) we have

$$\begin{aligned}
\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} |\overline{u}_\alpha(p')\Gamma u_\beta(p)|^2 &= \Gamma_{ab}(\not{p}+m)_{bc}\widetilde{\Gamma}_{cd}(\not{p}'+m)_{da} \\
&= [\Gamma(\not{p}+m)\widetilde{\Gamma}(\not{p}'+m)]_{aa} \\
&= \text{Tr}[\Gamma(\not{p}+m)\gamma^0\Gamma^\dagger\gamma^0(\not{p}'+m)] \quad , \tag{4.15}
\end{aligned}$$

which is the result in Eq. (4.10).

Analogous results hold for spin sums involving v spinors. For instance,

$$\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} |\overline{u}_\alpha(p')\Gamma v_\beta(p)|^2 = \text{Tr}\left[\Gamma(\not{p}-m)\gamma^0\Gamma^\dagger\gamma^0(\not{p}'+m)\right] \quad . \tag{4.16}$$

4.4 Elastic $e\mu$ scattering cross section

By using the general result (4.10), the matrix element (4.9) can be written as

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{4}\frac{e^4}{(q^2)^2}\text{Tr}\left[\gamma^\alpha\,(\not{k}'+m)\,\gamma^\lambda\,(\not{k}+m)\right]\text{Tr}\left[\gamma^\beta\,(\not{p}'+M)\,\gamma^\rho\,(\not{p}+M)\right]g_{\alpha\beta}\,g_{\lambda\rho} \quad . \tag{4.17}$$

We can in general evaluate traces of products of γ matrices by using the anticommutation relations (1.34) and (2.43). To do the calculation in Eq. (4.17) we need the following traces,

$$\text{Tr(odd number of } \gamma \text{ matrices}) = 0 \ , \tag{4.18}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu} \ , \tag{4.19}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4 \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} \right) \ , \tag{4.20}$$

which can be obtained using Eqs. (1.34),(2.43). By then carrying out the algebra in Eq. (4.17), we get

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{2e^4}{(q^2)^2} \left[2 \left(m^2 + M^2 \right) q^2 + (s - m^2 - M^2)^2 + (s + q^2 - m^2 - M^2)^2 \right] \ , \tag{4.21}$$

where s is the invariant center-of-mass energy square $s = (k+p)^2$.

We are now in a position to compute the cross section. This is given in terms of the scattering matrix element via Eq. (3.37),

$$d\sigma = \frac{1}{\mathcal{J}} |\mathcal{M}_{fi}|^2 \, d\Phi \ , \tag{4.22}$$

where $\mathcal{J}$ is the invariant initial-state flux and $d\Phi$ is the invariant final-state phase space. We can compute the cross section by plugging Eq. (4.21) into Eq. (4.22), choosing a reference frame, and evaluating the flux factor $\mathcal{J}$ and the phase space $d\Phi$ integration in this frame.

Consider the center-of-mass reference frame, $\mathbf{k} + \mathbf{p} = 0$ (Fig. 13). From Eq. (3.40) we have

$$\mathcal{J} = 4|\mathbf{p}|\sqrt{s} \ . \tag{4.23}$$

To carry out the integration over the final state phase space in Eq. (3.39),

$$d\Phi = \frac{d^3 k'}{(2\pi)^3 \, 2E'_k} \frac{d^3 p'}{(2\pi)^3 \, 2E'_p} (2\pi)^4 \delta^4(p' + k' - p - k) \ , \tag{4.24}$$

we can first use the three-momentum δ function to do the integral in $d^3 k'$, so that the cross section differential in the final-state solid angle $d\Omega = \sin\theta d\theta d\varphi$ can be written

$$\frac{d\sigma}{d\Omega} = \frac{1}{4|\mathbf{p}|\sqrt{s}} \frac{1}{(2\pi)^2} \int \mathbf{p}'^2 \, d|\mathbf{p}'| \, \frac{1}{4E'_p E'_k} \overline{|\mathcal{M}_{fi}|^2} \, \delta(E'_p + E'_k - E_p - E_k) \ . \tag{4.25}$$

Next it is convenient to make the change of integration variable

$$\begin{aligned} |\mathbf{p}'| \to E' &= \sqrt{\mathbf{p}'^2 + M^2} + \sqrt{\mathbf{p}'^2 + m^2} \\ &= E'_p + E'_k \ , \end{aligned} \tag{4.26}$$

with jacobian

$$\frac{\partial E'}{d|\mathbf{p}'|} = \frac{E'|\mathbf{p}'|}{E'_p E'_k} \ , \tag{4.27}$$

27

by which Eq. (4.25) can be rewritten as

$$\frac{d\sigma}{d\Omega} = \frac{1}{4|\mathbf{p}|\sqrt{s}} \frac{1}{(2\pi)^2} \int dE' \frac{|\mathbf{p}'|}{4E'} \overline{|\mathcal{M}_{fi}|^2} \, \delta(E' - \sqrt{s}) \ . \tag{4.28}$$

Performing the E' integral with the δ function and substituting the explicit expression (4.21) of $\overline{|\mathcal{M}_{fi}|^2}$, we obtain ($e^2 = 4\pi\alpha$)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2 \, s \, q^4} \left[2 \, (m^2 + M^2) \, q^2 + (s - m^2 - M^2)^2 + (s + q^2 - m^2 - M^2)^2 \right] \ . \tag{4.29}$$
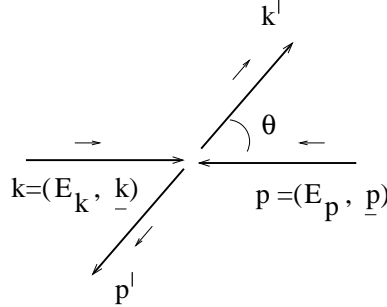
Figure 13: *Center-of-mass reference frame.*

The result (4.29) takes a simpler form in in the high energy limit $s \gg M^2, m^2$. In this case we have $s \to 4\mathbf{p}^2$, $q^2 \to -4\mathbf{p}^2 \sin^2(\theta/2)$, and the cross section becomes

$$\frac{d\sigma}{d\Omega} \simeq \frac{\alpha^2}{2 \, s \, \sin^4(\theta/2)} \left(1 + \cos^4 \theta/2\right) \quad \text{for} \quad s \gg M^2, m^2 \ . \tag{4.30}$$

The leading behavior of the cross section (4.30) at small angle is given by

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\theta^4} \quad \text{for} \quad \theta \ll 1 \quad , \tag{4.31}$$

where the θ^{-4} singularity is characteristic of the Coulomb interaction. It comes from the factor $1/(q^2)^2$, and reflects the long range of the interaction.

4.5 Scattering by an external Coulomb potential

The cross section for the scattering of a relativistic particle from an external Coulomb potential (Fig. 14) can be obtained as a particular case of the result of the previous subsection for $e\mu$ scattering, by working in the rest frame of μ and letting $M \to \infty$.

To this end, first express the invariant flux $\mathcal{J}$ in terms of the electron's three-momentum $\mathbf{k}$ in the rest frame of μ (Fig. 14),

$$\mathcal{J} = 4|\mathbf{k}|M \ . \tag{4.32}$$

Next, carry through the final-state phase space integration in terms of rest-frame variables. This yields

$$\frac{d\sigma}{d\Omega} = \frac{1}{4|\mathbf{k}|M} \frac{|\mathbf{k}'|}{16\pi^2 M} \overline{|\mathcal{M}_{fi}|^2} \ . \tag{4.33}$$
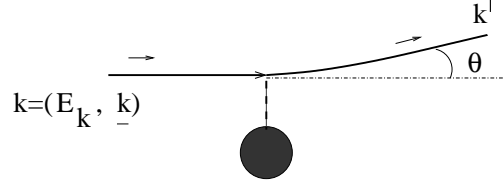
Figure 14: *Scattering by external Coulomb potential.*

Finally, let $M \to \infty$ in the invariant matrix element (4.21). In this limit $|\mathbf{k}'| \simeq |\mathbf{k}|$, $q^2 \simeq -4\mathbf{k}^2 \sin^2(\theta/2)$, and the square bracket in Eq. (4.21) becomes

$$\left[2\left(m^2 + M^2\right) q^2 + (s - m^2 - M^2)^2 + (s + q^2 - m^2 - M^2)^2\right]$$
$$\simeq \left[2M^2 q^2 + 2(2ME_k)^2 + \ldots\right]$$
$$\simeq 8M^2 E_k^2 \left[1 - (|\mathbf{k}|/E_k)^2 \sin^2(\theta/2)\right] \quad . \tag{4.34}$$

Substituting Eq. (4.34) into Eq. (4.33), we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4 \; \mathbf{k}^2 \; \mathbf{v}^2 \; \sin^4(\theta/2)} \left[1 - \mathbf{v}^2 \sin^2(\theta/2)\right] \quad , \tag{4.35}$$

where $\mathbf{v} = |\mathbf{k}|/E_k$.

Eq. (4.35) can be compared with the nonrelativistic result in Eq. (4.7). Observe that Eq. (4.35) has the form

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_R \left[1 - \mathbf{v}^2 \sin^2(\theta/2)\right] \quad , \tag{4.36}$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{\alpha^2}{4 \; \mathbf{k}^2 \; \mathbf{v}^2 \; \sin^4(\theta/2)} \tag{4.37}$$

is the Rutherford cross section, and the factor in the square bracket is the relativistic correction.

The relativistic correction $\left[1 - \mathbf{v}^2 \sin^2(\theta/2)\right]$ characterizes Coulomb scattering for spin 1/2. Computing Coulomb potential scattering of spinless charges, one finds

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{spin}-0} = \left(\frac{d\sigma}{d\Omega}\right)_R \quad , \tag{4.38}$$

that is, in the case of spinless particles the result does not differ from the nonrelativistic result. Eq. (4.36) shows that for $|\mathbf{v}| \to 1$ the angular distribution of a spin-1/2 particle differs from the nonrelativistic result as diffusion in the backward direction is strongly suppressed.
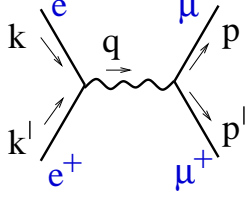
Figure 15: *Annihilation of electron pairs into muon pairs.*

4.6 Crossing symmetry: $e^+e^- \to \mu^+\mu^-$ annihilation

The $e\mu$ scattering process computed earlier is related by crossing symmetry to the annihilation process $e^+(k')e^-(k) \to \mu^+(p')\mu^-(p)$ (Fig. 15). Let us compute this in the approximation of massless electrons. The matrix element square, averaged over initial spins and summed over final spins, is given by

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{4} \frac{e^4}{(q^2)^2} \, \mathrm{Tr}\Big[\gamma^\rho \, (\not{p}' - M) \, \gamma^\sigma \, (\not{p} + M)\Big] \, \mathrm{Tr}\Big[\gamma^\tau \, \not{k}' \, \gamma^\lambda \, \not{k}\Big] \, g_{\lambda\rho} \, g_{\sigma\tau} \tag{4.39}$$

where $q^2 = s$, and we have set the electron mass to zero. Computing the traces yields

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{8e^4}{s^2} \, \Big[(p \cdot k)^2 + (p \cdot k')^2 + M^2 \, k \cdot k'\Big] \quad . \tag{4.40}$$

Let us work in the center-of-mass reference system, and denote by θ the center-of-mass scattering angle. In this system the matrix element square (4.40) takes the form

$$\overline{|\mathcal{M}_{fi}|^2} = e^4 \left[1 + \frac{4M^2}{s} + \left(1 - \frac{4M^2}{s}\right) \cos^2\theta\right] \quad . \tag{4.41}$$

The annihilation cross section can be computed via the general formula (3.37),

$$d\sigma = \frac{1}{\mathcal{J}} \, |\mathcal{M}_{fi}|^2 \, d\Phi \quad . \tag{4.42}$$

Note that for massless electrons

$$\mathcal{J} = 2s \quad , \tag{4.43}$$

and that the final-state phase space can be written as

$$d\Phi = \frac{|\mathbf{p}|}{16\pi^2\sqrt{s}} d\Omega \quad , \tag{4.44}$$

where $d\Omega = \sin\theta d\theta d\varphi$, and

$$|\mathbf{p}| = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4M^2}{s}} \quad . \tag{4.45}$$

Then the differential cross section $d\sigma/d\Omega$ is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^- \to \mu^+\mu^-} = \frac{\alpha^2}{4\,s} \sqrt{1 - \frac{4M^2}{s}} \left[1 + \frac{4M^2}{s} + \left(1 - \frac{4M^2}{s}\right) \cos^2\theta\right] \quad . \tag{4.46}$$

30

In the high energy limit $4M^2/s \to 0$, from Eq. (4.46) we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^-\to\mu^+\mu^-} = \frac{\alpha^2}{4\,s}\left(1+\cos^2\theta\right) \quad \text{for } s \gg M^2 \ . \tag{4.47}$$

The total cross section is obtained by integrating Eq. (4.46) over angles,

$$
\begin{aligned}
\sigma_{\text{tot}} &= \int \frac{d\sigma}{d\Omega} d\Omega \\
&= \frac{\alpha^2}{4\,s}\sqrt{1-\frac{4M^2}{s}}\, 2\pi\left[\left(1+\frac{4M^2}{s}\right)\int_0^\pi \sin\theta\,d\theta + \left(1-\frac{4M^2}{s}\right)\int_0^\pi \sin\theta\,\cos^2\theta\,d\theta\right] \\
&= \frac{4\,\pi\,\alpha^2}{3\,s}\sqrt{1-\frac{4M^2}{s}}\left(1+\frac{2M^2}{s}\right) \ . \tag{4.48}
\end{aligned}
$$

In the high energy limit,

$$\sigma_{\text{tot}} \simeq \frac{4\,\pi\,\alpha^2}{3\,s} \ . \tag{4.49}$$

Remark. In addition to the annihilation into muons and other leptons, experiments at high-energy e^+e^- accelerators measure the cross section for the annihilation of e^+e^- into hadrons. The cross section for $e^+e^- \to$ hadrons at large s differs from the expression (4.49) for $e^+e^- \to \mu^+\mu^-$ by a factor of the squared electric charge of the hadron constituents (quarks), summed over the possible N_c quark "colors" and N_f quark "flavors", and by effects of higher order in the strong interaction:

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\,\pi\,\alpha^2}{3\,s}\,N_c\sum_{i=1}^{N_f} Q_i^2\,[1+\mathcal{O}(\alpha_{\text{strong}})] \ . \tag{4.50}$$

5 Compton scattering

Let us analyze electron-photon Compton scattering at lowest order in e. This receives contribution from the graphs in Fig. 16.
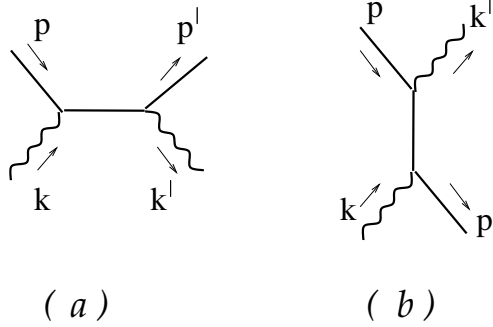

Figure 16: *Electron-photon scattering at lowest order in e.*

The scattering matrix element $\mathcal{M}_{fi}$ is given by

$$\mathcal{M}_{fi} = -ie^2 \varepsilon'^{\mu}(k')\varepsilon^{\nu}(k)\bar{u}(p') \left[\gamma_\mu \frac{(\not{p} + \not{k} + m)}{(p+k)^2 - m^2} \gamma_\nu + \gamma_\nu \frac{(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2} \gamma_\mu \right] u(p), \quad (5.1)$$

where $\varepsilon(k)$ and $\varepsilon'(k')$ are the incoming and outgoing photon polarization vectors. The two terms in the square bracket in Eq. (5.1) correspond respectively to graphs (a) and (b) in Fig. 16.

The sum of graphs (a) and (b) is gauge invariant, namely, with $\mathcal{M}_{fi} = M_{\mu\nu}\varepsilon'^{\mu}(k')\varepsilon^{\nu}(k)$, we have

$$M_{\mu\nu}k'^{\mu} = M_{\mu\nu}k^{\nu} = 0 \quad . \tag{5.2}$$

This can be seen by writing

$$
\begin{aligned}
M_{\mu\nu}k^{\nu} &= -ie^2\bar{u}(p') \left[\gamma_\mu \frac{1}{\not{p} + \not{k} - m} \not{k} + \not{k} \frac{1}{\not{p}' - \not{k} - m} \gamma_\mu \right] u(p) \\
&= -ie^2\bar{u}(p') \left\{ \gamma_\mu \frac{1}{\not{p} + \not{k} - m} [(\not{p} + \not{k} - m) - (\not{p} - m)] \right. \\
&\quad + \left. [-(\not{p}' - \not{k} - m) + (\not{p}' - m)] \frac{1}{\not{p}' - \not{k} - m} \gamma_\mu \right\} u(p) \quad . \tag{5.3}
\end{aligned}
$$

Then, from each of the square brackets in the last two lines, the contributions of the first terms cancel each other, while the contributions of the second terms vanish separately because of the Dirac equation. Thus

$$
\begin{aligned}
M_{\mu\nu}k^{\nu} &= -ie^2\bar{u}(p') \left[\gamma_\mu \frac{1}{\not{p} + \not{k} - m} (\not{p} - m) + (\not{p}' - m) \frac{1}{\not{p}' - \not{k} - m} \gamma_\mu \right] u(p) \\
&= 0 \quad . \tag{5.4}
\end{aligned}
$$

Similarly for the dot product of $M_{\mu\nu}$ with k'^{μ}.

To compute the unpolarized cross section, we need the squared matrix element, averaged over the initial electron and photon polarizations, and summed over the final electron and photon polarizations,

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{4} \sum_{polarizations} |\mathcal{M}_{fi}|^2 \ . \tag{5.5}$$

The electron sums can be dealt with using the general result in Subsec. 4.3. The photon sums are discussed in the next subsection.

5.1   Photon polarization sums

The sum over photon polarizations can be performed by replacing the sum with $-g^{\mu\nu}$,

$$\sum_{\alpha=1}^{2} \varepsilon_\mu^\alpha(k)\varepsilon_\nu^\alpha(k) \to -g_{\mu\nu} \ , \tag{5.6}$$

because the amplitude into which ε is dotted is conserved, Eq. (5.2).

To see this, consider a matrix element of the form

$$A^\mu \varepsilon_\mu(k) \tag{5.7}$$

for A^μ such that

$$A^\mu k_\mu = 0 \ , \tag{5.8}$$

and sum the matrix element square over polarizations,

$$\sum_{\alpha=1}^{2} |A^\mu \varepsilon_\mu^\alpha(k)|^2 = \sum_{\alpha=1}^{2} A^\mu A^\nu \varepsilon_\mu^\alpha(k)\varepsilon_\nu^\alpha(k) \ . \tag{5.9}$$

Now use that polarizations form an orthonormal set in the plane transverse to the momentum $\mathbf{k}$,

$$\sum_{\alpha=1}^{2} \varepsilon^{\alpha i}(k)\varepsilon^{\alpha j}(k) = \delta^{ij} - \hat{k}^i\hat{k}^j \ , \quad \text{where} \quad \hat{k}^i = k^i/|\mathbf{k}| = k^i/k^0 \ . \tag{5.10}$$

Then the sum (5.9) can be written

$$
\begin{aligned}
\sum_{\alpha=1}^{2} |A^\mu \varepsilon_\mu^\alpha(k)|^2 &= \sum_{\alpha=1}^{2} A^\mu A^\nu \varepsilon_\mu^\alpha(k)\varepsilon_\nu^\alpha(k) \\
&= A^i A^j (\delta^{ij} - \hat{k}^i\hat{k}^j) = A^i A^i - (A^i\hat{k}^i)(A^j\hat{k}^j) \\
&= A^i A^i - A^0 A^0 = -A^\mu A^\nu g_{\mu\nu} \ ,
\end{aligned}
\tag{5.11}
$$

where in the last line we have used that Eq. (5.8) implies

$$A^0 k^0 - A^i k^i = 0 \ , \quad i.e., \quad A^i\hat{k}^i = A^0 \ . \tag{5.12}$$

From Eqs. (5.9) and (5.11) we obtain that the sum over polarizations amounts to

$$\sum_{\alpha=1}^{2} \varepsilon_\mu^\alpha(k)\varepsilon_\nu^\alpha(k) \to -g_{\mu\nu} \ , \tag{5.13}$$

as stated in Eq. (5.6).

5.2 The $e\gamma$ unpolarized cross section

The unpolarized matrix element square (5.5) can now be determined from Eq. (5.1) by using the result in Eq. (4.10) for the electron polarization sums and the result in Eq. (5.6) for the photon polarization sums.

To do this calculation, we need to evaluate the traces of γ matrices generated by the electron spin sums in Eq. (4.10), making use of the γ matrix identities

$$\gamma_\mu \gamma^\mu = 4 \ , \tag{5.14}$$

$$\gamma_\mu \gamma^\rho \gamma^\mu = -2\gamma^\rho \ , \tag{5.15}$$

which follow from the anticommutation relations (1.34). By working out the algebra, we obtain the result

$$
\begin{aligned}
\overline{|\mathcal{M}_{fi}|^2} &= \frac{1}{4} \sum_{polarizations} |\mathcal{M}_{fi}|^2 \\
&= 2e^4 \left[\frac{p\cdot k}{p\cdot k'} + \frac{p\cdot k'}{p\cdot k} + 2m^2 \left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right) + m^4 \left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right)^2 \right]
\end{aligned} \tag{5.16}
$$

The $e\gamma$ cross section is related to the scattering matrix element via the general formula (3.37),

$$d\sigma = \frac{1}{\mathcal{J}} |\mathcal{M}_{fi}|^2 \, d\Phi \ . \tag{5.17}$$

We can compute it by choosing a reference frame, plugging Eq. (5.16) into Eq. (5.17) and evaluating the flux factor $\mathcal{J}$ and phase space $d\Phi$ integration.
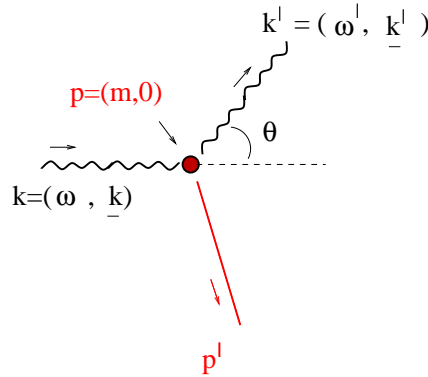

Figure 17: *Compton scattering in the laboratory frame.*

Consider the laboratory frame in which the electron is initially at rest, $p^\mu = (m, \mathbf{0})$ (Fig. 17). In the notation of Fig. 17 we have

$$m^2 = p'^2 = (p + k - k')^2 = m^2 + 2m(\omega - \omega') - 2\omega\omega'(1 - \cos\theta) \ , \tag{5.18}$$

which gives

$$\omega' = \frac{\omega}{1 + (\omega/m)(1 - \cos\theta)} \ . \tag{5.19}$$

By evaluating the right hand side of Eq. (5.16) in the laboratory frame and using Eq. (5.19), we obtain

$$\overline{|\mathcal{M}_{fi}|^2} = 2e^4 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \ . \tag{5.20}$$

From Eq. (3.40) for the flux factor we get

$$\mathcal{J} = 4E_a E_b |v_a - v_b| = 4m\omega \ . \tag{5.21}$$

Let us now carry out the integration over the final-state phase space (3.39),

$$d\Phi = \frac{d^3 k'}{(2\pi)^3 \, 2\omega'} \frac{d^3 p'}{(2\pi)^3 \, 2E'} \ (2\pi)^4 \delta^4(p' + k' - p - k) \ . \tag{5.22}$$

By using the three-momentum δ function to perform the integral in d^3p', and inserting the results (5.20) and (5.21) into Eq. (5.17), the differential cross section in the solid angle Ω of the final photon momentum is given by

$$\frac{d\sigma}{d\Omega} = \frac{2e^4}{4m\omega} \int d\omega' \frac{\omega'}{16\pi^2 E'} \ \delta(E' + \omega' - m - \omega) \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right] \ , \tag{5.23}$$

where

$$E' = \sqrt{\omega^2 + \omega'^2 - 2\omega\omega' \cos \theta + m^2} \ . \tag{5.24}$$

Performing the integral in $d\omega'$, we arrive at the unpolarized electron-photon cross section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2} \left(\frac{\omega'}{\omega} \right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right] \ , \tag{5.25}$$

where $\alpha = e^2/4\pi$.

In the low energy limit $\omega \ll m$, from Eq. (5.19) we have $\omega' \approx \omega$, and Eq. (5.25) reduces to the Thomson cross section,

$$\frac{d\sigma}{d\Omega} \longrightarrow \frac{\alpha^2}{2m^2} \ (1 + \cos^2 \theta) \qquad \text{for } \omega \ll m \ , \tag{5.26}$$

describing the scattering of classical electromagnetic radiation by a free electron.

In the high energy limit $\omega \gg m$, Eq. (5.25) gives rise to a logarithmic behavior in the total cross section, arising from the emission of photons at small angles. This is because for $\omega \gg m$ from Eq. (5.19) we have

$$\omega' \simeq \frac{m}{1 - \cos \theta} \quad \text{for} \quad \frac{\omega}{m} \ (1 - \cos \theta) \gg 1 \ , \tag{5.27}$$

which means that in the region

$$1 \gg \theta^2 \gg \frac{2m}{\omega} \tag{5.28}$$

the cross section is strongly peaked,

$$\frac{d\sigma}{d\Omega} \ \simeq \ \frac{\alpha^2}{2m^2} \left(\frac{m}{\omega} \right)^2 \frac{1}{(1 - \cos \theta)^2} \left[\frac{\omega}{m} \ (1 - \cos \theta) - \sin^2 \theta \right]$$

$$\simeq \ \frac{\alpha^2}{2m\omega} \frac{1}{1 - \cos \theta} \ . \tag{5.29}$$

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Integrating over angles gives

$$\sigma \simeq 2\pi \int d\cos\theta \; \frac{\alpha^2}{2m\omega} \frac{1}{1-\cos\theta}$$

$$\simeq 2\pi \int_{2m/\omega}^1 \frac{d\theta^2}{\theta^2} \frac{\alpha^2}{m\omega} \simeq \frac{2\pi\alpha^2}{m\omega} \ln\frac{\omega}{2m} \;. \tag{5.30}$$

The total cross section σ falls like ω^{-1}, but with a logarithmic enhancement from the integration over the small-angle, or collinear, region. The collinear region is cut off by the mass m. The occurrence of collinear logarithms illustrated by this example is a general feature associated with the massless limit of the theory.

5.3   Photon polarization dependence

The calculation performed above can be redone for fixed $\varepsilon(k)$ and $\varepsilon'(k')$ to obtain the dependence of the cross section on the initial and final photon polarizations. The result for the cross section including the polarization dependence is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol.}} = \frac{\alpha^2}{4m^2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 4(\varepsilon \cdot \varepsilon')^2 - 2\right] \;. \tag{5.31}$$

From Eq. (5.31) we recover Eq. (5.25) through averaging over ε and summing over ε' by using Eq. (5.10), i.e., that the sum over polarizations gives the transverse projector with respect to the momentum,

$$\sum_{\alpha=1}^{2} \varepsilon_i^\alpha(k)\varepsilon_j^\alpha(k) = \delta_{ij} - \hat{k}_i\hat{k}_j \quad, \tag{5.32}$$

$$\sum_{\beta=1}^{2} \varepsilon_i'^\beta(k')\varepsilon_j'^\beta(k') = \delta_{ij} - \hat{k}_i'\hat{k}_j' \quad. \tag{5.33}$$

We thus have

$$\frac{1}{2} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \left(\frac{d\sigma}{d\Omega}\right)_{\text{pol.}}$$

$$= \frac{\alpha^2}{4m^2} \left(\frac{\omega'}{\omega}\right)^2 \left\{2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2\right] + 4 \frac{1}{2} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} [\varepsilon^\alpha(k) \cdot \varepsilon'^\beta(k')]^2 \right\} \;, \tag{5.34}$$

where

$$\frac{1}{2} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} [\varepsilon^\alpha(k) \cdot \varepsilon'^\beta(k')]^2 = \frac{1}{2} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \varepsilon_i^\alpha(k)\varepsilon_j^\alpha(k)\varepsilon_i'^\beta(k')\varepsilon_j'^\beta(k')$$

$$= \frac{1}{2} (\delta_{ij} - \hat{k}_i\hat{k}_j)(\delta_{ij} - \hat{k}_i'\hat{k}_j') = \frac{1}{2} [1 + (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')^2]$$

$$= \frac{1}{2} (1 + \cos^2\theta) = \frac{1}{2} (2 - \sin^2\theta) \;. \tag{5.35}$$

Substituting Eq. (5.35) into Eq. (5.34) we re-obtain the unpolarized result (5.25).

6 Strong interactions

In this section we extend the discussion given in the previous sections to the case of strong interactions. The theory of strong interactions, Quantum Chromodynamics (QCD), is treated systematically in the Standard Model course. Here we give a brief introduction building on material presented for QED. In Subsec. 6.1 we introduce the gauge symmetry of the strong interaction as a generalization of that of QED and examine the couplings that follow from it. In Subsec. 6.2 we contrast features of photon and gluon polarization degrees of freedom and discuss physical implications. In Subsec. 6.3 we give basic results on the algebra of QCD charges that serve in practical calculations.

We will use the results presented here to further discuss strong interactions in Sec. 8.

6.1 Basic structure

We can think of QCD as a theory similar to QED but with

- $N = 3$ charged spin-1/2 particles ψ_i (quarks, replicated in six families — the so-called quark "flavors"),

- $N^2 - 1 = 8$ gauge bosons A_μ^a (gluons),

with different couplings to different charges. The couplings are to be thought of as matrices

$$T^a \ , \quad a = 1, \ldots, N^2 - 1 \tag{6.1}$$

obeying well-prescribed commutation relations

$$[T^a, T^b] = if^{abc}T^c \ , \tag{6.2}$$

where f^{abc} are "structure constants", antisymmetric in all indices. The quantum number specifying the charge of QCD is called "color", and T^a are the color-charge matrices. Thus QCD contains multiple vector particles and its charges are non-commuting, or "non-abelian".

Eq. (6.2) defines an algebra of color-charge operators whose formal properties can usefully be thought of along similar lines to the discussion given in Sec. 2 for the algebra (2.1) of angular momentum operators J^i. J^i are the generators of the rotation group. T^a are the generators of the color symmetry group ($SU(N)$ with $N = 3$), and have matrix representations for different dimensionalities n. The fundamental representation is the representation with dimensionality $n = N$ to which quarks belong, $\psi_i, i = 1, 2, 3$. The matrix representation of the generators is given by

$$T^a \to \frac{1}{2} \lambda^a \ , \tag{6.3}$$

where λ^a are the eight Gell-Mann 3×3 matrices. The adjoint representation is the representation with dimensionality $n = N^2 - 1$ to which gluons belong, $A_\mu^a, a = 1, \ldots, 8$. The matrix representation of the generators in the adjoint is given by the structure constants themselves,

$$(T^a)^{bc} \to -if^{abc} \ . \tag{6.4}$$

As in QED, the spin-1/2 charged particles satisfy equations of motion of Dirac type,

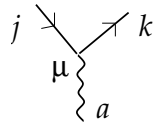$$(i\partial\!\!\!/ - m)\psi = 0 \ , \tag{6.5}$$

and we can write down their coupling to the vector particles, the gluons, based on similar reasoning as in QED. While in QED we write the electron-photon interaction by replacing

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu \qquad \text{(QED)} \tag{6.6}$$

in the equations of motion, in QCD it is not just one term by which we modify ∂^μ but a sum of terms, one for each of the gluons, and each term is proportional not just to a number like the electric charge but to a color-charge matrix:

$$\partial_\mu \to D_\mu = \partial_\mu + ig_s A_\mu^a T^a \qquad \text{(QCD)} \ , \tag{6.7}$$
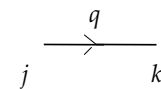
where g_s is the strong-interaction coupling constant. With this interaction term, by going through the analogous perturbation analysis as for QED, we can extract the Feynman rule for the quark-quark-gluon coupling. This is given in Fig. 18. It has a similar structure to the QED vertex of Fig. 6, with the difference being in the color matrix.
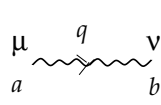

$$i g_s \ \gamma^\mu \ (T^a)_{jk}$$

Figure 18: *QCD Feynman rule for quark-quark-gluon vertex.*

For internal lines we have the Feynman rules for propagators (Fig. 19), similar to those of QED except for the additional dependence on color indices.

$$\delta^{jk} \ \frac{i \ (q\!\!\!/ + m)}{q^2 - m^2 + i\varepsilon}$$

$$\delta^{ab} \ \frac{-ig^{\mu\nu}}{q^2 + i\varepsilon} \qquad \text{(Feynman gauge)}$$

Figure 19: *QCD Feynman rules for quark and gluon propagators.*

The quark-gluon coupling above does not exhaust QCD interactions, though. In a theory of multiple vector particles such as QCD the vector particles turn out to necessarily be self-interacting. The reason for this lies with the non-abelian nature of the color

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charges, i.e., with the nonzero commutator (6.2). The origin and precise form of the gluon self-couplings can specifically be traced back to the form of the gauge transformations in QCD and relationship between potentials and fields.

Electromagnetism is invariant under gauge transformations given by changes of the four-potential A_μ by an arbitrary four-gradient,

$$A_\mu \to A_\mu + \partial_\mu \lambda \qquad \text{(QED)} \quad . \tag{6.8}$$

In QCD the gauge freedom involves an additional contribution, namely we can change A_μ by a four-gradient and/or by a pure rotation of its color indices,

$$A_\mu^a \to A_\mu^a + \partial_\mu \lambda^a - g_s f^{abc} \lambda^b A_\mu^c \qquad \text{(QCD)} \quad , \tag{6.9}$$

leaving physics invariant. Eq. (6.9) specifies the form of the gauge transformations in QCD. Because of the nonvanishing structure constants f^{abc}, while the field strength tensor $F_{\mu\nu}$ in QED is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \text{(QED)} \quad , \tag{6.10}$$

in QCD this acquires an extra term,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c \qquad \text{(QCD)} \quad . \tag{6.11}$$

It is precisely the extra term in $F_{\mu\nu}^a$ in Eq. (6.11) which is responsible for producing gluon self-interactions when one constructs a gauge-invariant kinetic energy term for gluons by taking the square of $F_{\mu\nu}^a$, analogously to the case of photons. The square of $F_{\mu\nu}^a$ in Eq. (6.11) gives rise to both cubic and quartic gluon self-interaction terms. The cubic term is proportional to $g_s \times f$ and contains derivative couplings, while the quartic term is proportional to $g_s^2 \times f^2$, with no derivatives. Their precise form is given in Fig. 20.

Figure 20: *QCD Feynman rules for cubic and quartic gluon vertices.*

The construction above implies in particular that the coupling constant g_s in the gauge boson self-interaction vertices is one and the same as the coupling constant g_s in the quark-gluon interaction vertex. Later in the section we see a specific example showing that this equality of couplings is necessary for non-abelian gauge invariance to be satisfied.

6.2 Physical polarization states and ghosts

In this subsection we discuss implications of the non-abelian gauge symmetry by comparing features of photon and gluon polarization degrees of freedom. We start from examining gauge invariance in a simple example, the QCD analogue of Compton scattering, and contrast the case of QCD with the case of QED seen in Sec. 5.
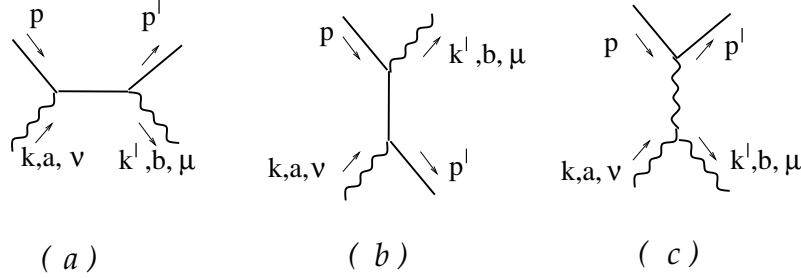

Figure 21: *Quark-gluon Compton scattering at lowest order in g_s.*

The QCD analogue of Compton scattering is the quark-gluon scattering depicted in Fig. 21 at lowest order in the coupling g_s. The graphs in Fig. 21(a) and (b) are analogous to the QED graphs of Fig. 16, while the three-gluon coupling graph in Fig. 21(c) is non-abelian. The scattering matrix element $\mathcal{M}_{fi}$ can be written as

$$\mathcal{M}_{fi} = M_{\mu\nu}\varepsilon'^{\mu}(k')\varepsilon^{\nu}(k) \ . \tag{6.12}$$

From graphs (a) and (b) in Fig. 21 we have

$$M_{\mu\nu}^{(a)+(b)} = -ig_s^2\bar{u}(p')\left[\gamma_\mu T^b \frac{1}{\not{p}+\not{k}-m} \gamma_\nu T^a + \gamma_\nu T^a \frac{1}{\not{p}-\not{k'}-m} \gamma_\mu T^b\right]u(p) \ . \tag{6.13}$$

The sum of graphs (a) and (b) is not by itself gauge-invariant, because by dotting Eq. (6.13) into k^ν we get

$$M_{\mu\nu}^{(a)+(b)} \ k^\nu = ig_s^2[T^a,T^b]\bar{u}(p')\gamma_\mu u(p) \ , \tag{6.14}$$

which is nonvanishing due to the nonzero commutator of color charges.

From graph (c) in Fig. 21 we have

$$M_{\mu\nu}^{(c)} = g_s^2\bar{u}(p')\gamma^\rho T^c \ u(p)\frac{1}{(k-k')^2} \ f^{abc}\left[g_{\nu\mu}(k+k')_\rho + g_{\mu\rho}(k-2k')_\nu + g_{\rho\nu}(k'-2k)_\mu\right]. \tag{6.15}$$

By dotting Eq. (6.15) into k^ν we get

$$\begin{aligned}
M_{\mu\nu}^{(c)} \ k^\nu &= g_s^2 f^{abc}T^c\bar{u}(p')\gamma_\mu u(p) + \{\ldots\} \ k'_\mu \\
&= -ig_s^2[T^a,T^b]\bar{u}(p')\gamma_\mu u(p) + \{\ldots\} \ k'_\mu
\end{aligned} \tag{6.16}$$

where in the last line we have used Eq. (6.2) to rewrite $f^{abc}T^c$ in terms of the commutator. The second term in the right hand side of Eq. (6.16) is a term proportional to k'_μ, which

gives zero once it is dotted into physical polarizations $\varepsilon'(k') \cdot k' = 0$. The first term, on the other hand, precisely cancels the contribution in Eq. (6.14). Thus gauge invariance is achieved once graph (c) is added to graphs (a) and (b). This illustrates that gauge boson self-interactions are required by gauge invariance in the non-abelian case, and that their coupling constant must equal that of quark-gluon interactions.

The term in k'_μ in the right hand side of Eq. (6.16), however, signifies that gauge invariance is realized in quite a different manner than in the abelian case. In Eq. (6.16) we obtain $M_{\mu\nu}k^\nu = 0$ only if μ is restricted to physical polarizations, while in the QED case, Eq. (5.4), we have $M_{\mu\nu}k^\nu = 0$ regardless of μ. While this is of no consequence in the present lowest-order case, since we are entitled to enforce physical gluon polarizations, it implies a profound difference when we analyze the theory beyond lowest order and include loops (as we will do in the next two sections).

Recall that in QED as a consequence of $M_{\mu\nu}k^\nu = 0$ we arrived at the equivalence implied by Eq. (5.6),

$$\sum_{\alpha=1}^{2} \varepsilon_\mu^\alpha(k)\varepsilon_\nu^\alpha(k) \to -g_{\mu\nu} \quad \text{(QED)} \quad , \tag{6.17}$$

in which the sum on the left hand side is over transverse polarizations, while the right hand side sums over all covariant polarization states, including the unphysical longitudinal ones. This means that the structure of the abelian theory implies that unphysical polarization states automatically cancel. The result we have just found for QCD indicates that this cancellation is not automatic in the non-abelian theory. Thus, if we are to restore the equivalence between sum over physical states and sum over covariant states in the QCD case, further degrees of freedom are to be added in to the theory, which will have to be such that they cancel the contribution of unphysical gluon polarizations.
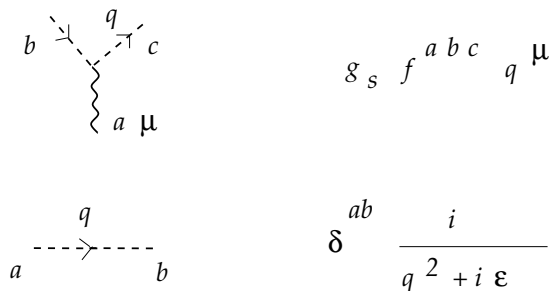

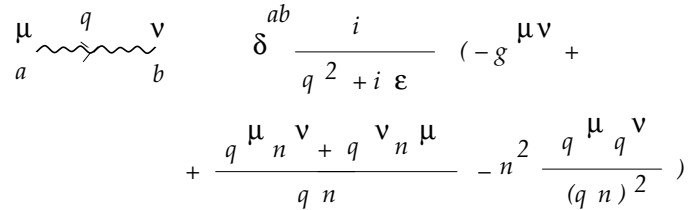Figure 22: *QCD Feynman rules for ghost vertex and ghost propagator.*

The systematic construction of the theory shows that such terms emerge in a well-prescribed, precise manner. They are referred to as ghosts and correspond to well-defined, but not physical, degrees of freedom, which can propagate and couple to gluons, but never be produced in physical final states. Their role is precisely that of canceling unphysical gluon polarization states. The Feynman rules for ghost propagator and coupling are given in Fig. 22. Ghosts transform under the adjoint representation (6.4) of the color symmetry group, and transform like scalars under Lorentz (though they obey anticommuting relations like fermions), hence the form of their propagator and deriva-

tive coupling in Fig. 22. Ghosts are a non-abelian effect. Their coupling is proportional to f^{abc}. We could introduce ghosts in electrodynamics, but they will just decouple.

It is possible to exploit the gauge freedom in order to devise a gauge-fixing condition such that the unphysical gluon polarizations are automatically eliminated, and thus no need arises for ghosts. Such gauges without ghosts are referred to as physical gauges, and are the gauges in which the non-abelian theory looks the most like its abelian counterpart. They are distinct from the covariant gauges based on the Lorentz gauge-fixing condition (3.12) in which we have worked so far. The gauge-fixing relation for physical gauges is given by assigning a four-vector n^μ and setting

$$A^a_\mu \, n^\mu = 0 \ . \tag{6.18}$$

Physical gauges (6.18) present certain advantages, as they do not contain unphysical degrees of freedom. However, the form of the gluon propagator becomes rather more complicated in these gauges. This is given in Fig. 23.

Figure 23: *Gluon propagator in physical gauge $A \cdot n = 0$.*

6.3 Color algebra

QCD calculations involve charge factors based on the color algebra in Eq. (6.2). We here introduce basic invariants of the algebra which occur in practical applications, the Casimir invariants C_R and the trace invariants T_R. The subscript R specifies the representation of the algebra (6.2).

The Casimir invariant can be defined from the square operator $T^2 = T^a T^a$. This operator commutes with all generators of the algebra (6.2),

$$\begin{aligned}
[T^a T^a, T^b] &= T^a[T^a, T^b] + [T^a, T^b]T^a \\
&= i f^{abc}(T^a T^c + T^c T^a) = 0 \tag{6.19}
\end{aligned}$$

where the last line vanishes due to f^{abc} being antisymmetric. This is analogous to the case of the algebra of angular momentum operators, where $[J^2, J^i] = 0$, and the eigenvalue of the square operator J^2 is used to label different representations. In the case of the color charge operators, Eq. (6.19) implies that T^2 takes a constant value on each representation, and the matrix representation of T^2 is given by a constant C_R times the identity matrix,

$$T^a T^a = C_R \, \mathbf{1} \ . \tag{6.20}$$

Here $\mathbf{1}$ is the identity matrix in n dimensions, where n is the dimensionality of the representation. The constant C_R is the Casimir invariant, and characterizes the specific representation.

The trace invariant can be defined from the trace of two generators, $\text{Tr}(T^a T^b)$, based on the fact that one can choose a basis such that this trace is proportional to δ^{ab},

$$\text{Tr}(T^a T^b) = T_R \ \delta^{ab} \ . \tag{6.21}$$

The constant T_R is the trace invariant, specific to any given representation. Conventionally we normalize the color generators so that in the fundamental representation $R = F$, specified by the generator matrices (6.3), we have

$$T_F = \frac{1}{2} \ . \tag{6.22}$$

Once this is fixed, all other Casimir and trace invariants are determined in all representations. The normalization (6.22) for the color generators is analogous to that of the angular momentum generators in the spin-1/2 representation (2.3), $J^i \to \sigma^i/2$, for which

$$\text{Tr}(J^i J^j) = \frac{1}{2} \ \delta^{ij} \ . \tag{6.23}$$

The Casimir invariant C_R and the trace invariant T_R are related to each other, because if we multiply Eq. (6.21) by δ^{ab} we get

$$\delta^{ab} \ \text{Tr}(T^a T^b) = T_R \ \delta^{ab} \ \delta^{ab} = T_R \ d \ , \tag{6.24}$$

where $d = N^2 - 1$ is the number of generators, and therefore, using Eq. (6.20) to evaluate the left hand side of Eq. (6.24), we have

$$C_R \ n = T_R \ d \ . \tag{6.25}$$

Eq. (6.25) implies that the Casimir invariant for the fundamental representation $R = F$, for which $n = N = 3$, is given by

$$C_F = T_F \ \frac{d}{n} = \frac{N^2 - 1}{2N} = \frac{4}{3} \ , \tag{6.26}$$

where we have used Eq. (6.22). In the adjoint representation $R = A$ specified by the generator matrices (6.4), for which $n = N^2 - 1$, Eq. (6.25) implies that the Casimir invariant and trace invariant are equal, $C_A = T_A$. By performing the explicit calculation we get

$$C_A = T_A = N = 3 \ . \tag{6.27}$$

We will see examples of QCD calculations involving the color factors above in Sec. 8.

7 Renormalization

So far we have considered tree-level effects, that is, Feynman graphs that do not contain loops. Loop corrections to processes described in earlier sections arise at higher orders of perturbation theory. For instance, the graph in Fig. 24 is an example of a loop correction to fermion pair production.
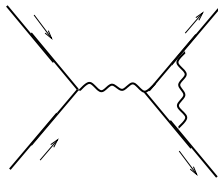

Figure 24: *Loop correction to fermion pair production.*

In this section we address loop effects. The part of the theory that deals with these effects is renormalization. We give an introduction to the idea of renormalization and its physical implications, discussing two specific examples: i) the renormalization of the electric charge; ii) the electron's anomalous magnetic moment. In Sec. 8 we continue the discussion by introducing further concepts and including both electromagnetic and strong interactions.

7.1 General principles

While the treatment given so far specifies interaction processes at tree level, the method of renormalization is required to treat processes including loops.

A symptom that renormalization is required is that Feynman graphs with loops may give rise to integrals containing divergences from high-momentum regions. Renormalization allows one to give meaning to the occurrence of these ultraviolet divergences.

Ultraviolet power counting provides the basic approach to renormalization. For a Feynman graph involving a loop integral of the form

$$\int d^4k \; \frac{N(k)}{M(k)} \; , \tag{7.1}$$

consider the superficial degree of divergence defined as

$$D = (powers \; of \; k \; in \; N \; + \; 4) \; - \; (powers \; of \; k \; in \; M) \; . \tag{7.2}$$

If $D \geq 0$, the integral is ultraviolet divergent. A first way of characterizing a theory as "renormalizable" is that the number of ultraviolet divergent amplitudes is finite. This is the case for instance with QED. There are 3 ultraviolet divergent amplitudes in QED, depicted in Fig. 25. QCD has a few more, due to the more complex structure of interactions seen in Sec. 6, but still a finite number. In a renormalizable theory there can of course be infinitely many Feynman graphs that are ultraviolet divergent, but they are so because they contain one of the few primitively divergent amplitudes as a subgraph.
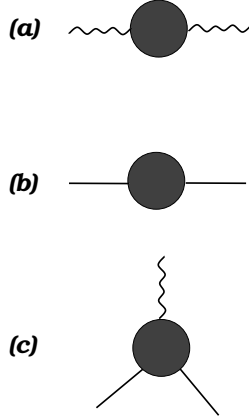
Figure 25: *Ultraviolet divergent amplitudes in QED: (a) photon self-energy; (b) electron self-energy; (a) electron-photon vertex.*

The main point about renormalizability is that it implies that all the ultraviolet divergences can be absorbed, according to a well-prescribed procedure specified below, into rescalings of the parameters and wave functions in the theory. For a given quantity ϕ, the rescaling is of the form

$$\phi \to \phi_0 = Z\,\phi \ , \tag{7.3}$$

where ϕ_0 and ϕ are respectively the unrenormalized and renormalized quantities, and Z is a calculable constant, into which the divergence can be absorbed. Here Z is the renormalization constant, possibly divergent but unobservable. Once the rescalings are done and the predictions of the theory are expressed in terms of renormalized quantities, all physical observables are finite and free of divergences.

This leads to a characterization of the renormalization program which we can formulate as a sequence of steps as follows.

- Compute the divergent amplitudes, by prescribing a "regularization method". Examples of regularization methods are a cut-off Λ on the ultraviolet integration region, where the result diverges as we let $\Lambda \to \infty$, or, as we will see in explicit calculations later, dimensional regularization.

- Assign parameter and wave-function rescalings to eliminate divergences. In the case of QED, these involve the electromagnetic potential A, the electron wave function ψ and mass m, and the coupling e. Using traditional notation for the QED renormalization constants Z_i, the rescalings can be written as

$$
\begin{aligned}
A \to A_0 &= \sqrt{Z_3}\,A \ , \\[2mm]
\psi \to \psi_0 &= \sqrt{Z_2}\,\psi \ , \\[2mm]
m \to m_0 &= \frac{Z_m}{Z_2}\,m \ , \\[2mm]
e \to e_0 &= \frac{Z_1}{Z_2\sqrt{Z_3}}\,e \ .
\end{aligned}
\tag{7.4}
$$

Here Z_3 and Z_2 are the respectively the renormalization constants for the photon and electron wave function, Z_1 is the vertex renormalization constant and Z_m is the electron mass renormalization constant.

- Once the rescalings are done, all physical observables are calculable, i.e., unambiguously defined in terms of renormalized quantities, and free of divergences.

Theories for which this program succeeds in giving finite predictions for physical quantities are renormalizable theories. Non-renormalizable theories are theories in which one cannot absorb all divergences in a finite number of Z: for instance, as we go to higher orders new divergences appear and an infinite number of Z is needed.

The above program, while it appears quite abstract at first, gives in fact testable, measurable effects. In the next few subsections we see specific examples of this.

A further, general point is that gauge invariance places strong constraints on renormalization, implying relations among the divergent amplitudes of the theory, and thus among the renormalization constants. Here is an example for the case of QED. Gauge invariance establishes the following relation between the electron-photon vertex Γ_μ dotted into the photon momentum q^μ and the electron propagators S,

$$q^\mu \Gamma_\mu = S^{-1}(p+q) - S^{-1}(p) \ . \tag{7.5}$$

Eq. (7.5), pictured in Fig. 26, is referred to as the Ward identity and is valid to all orders. Using the renormalization constants Z_1 and Z_2 defined by the rescalings in Eq. (7.4),

$$\Gamma^\mu = \frac{1}{Z_1} \gamma^\mu + \dots \ , \qquad S(p) = \frac{Z_2}{\not{p} - m} + \dots \ , \tag{7.6}$$

we have

$$\frac{1}{Z_1} \not{q} = \frac{1}{Z_2} \left[(\not{p} + \not{q} - m) - (\not{p} - m) \right] \ . \tag{7.7}$$

Thus in the abelian case

$$Z_1 = Z_2 \qquad \text{(QED)} \ . \tag{7.8}$$

As a result, the rescaling relation in Eq. (7.4) defining the renormalized coupling in QED becomes

$$e^2 = Z_3 e_0^2 \ . \tag{7.9}$$

That is, the renormalization of the electric charge is entirely determined by the renormalization constant Z_3, associated with the photon wave function, and does not depend on any other quantity related to the electron.
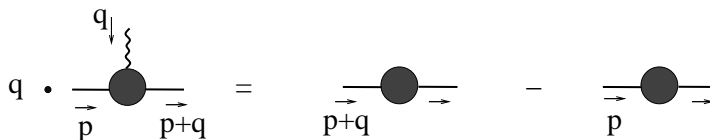

Figure 26: *Relation between electron-photon vertex and electron propagators.*

In the non-abelian case the relation (7.8) does not apply. However, it is still valid that non-abelian gauge invariance sets constraints on renormalization, leading to other, more complex relations among the renormalization constants. We will see examples of this in Sec. 8.3.

In the rest of this section we describe specific calculations of renormalization at one loop.

7.2   The gauge boson self-energy

Let us consider the gauge boson self-energy. This is one of the divergent amplitudes shown in Fig. 25. The Feynman graphs contributing to the self-energy through one loop are given in Fig. 27 for the photon and gluon cases. In the photon case one has the fermion loop graph only, while in the gluon case one has in addition gluon loop and ghost loop graphs.

Because of the relations (7.8),(7.9), in the QED case the calculation of the gauge boson self-energy is all that is needed to determine the renormalization of the coupling. So the result of this subsection will be used in Subsec. 7.3 to discuss the renormalized electric charge.
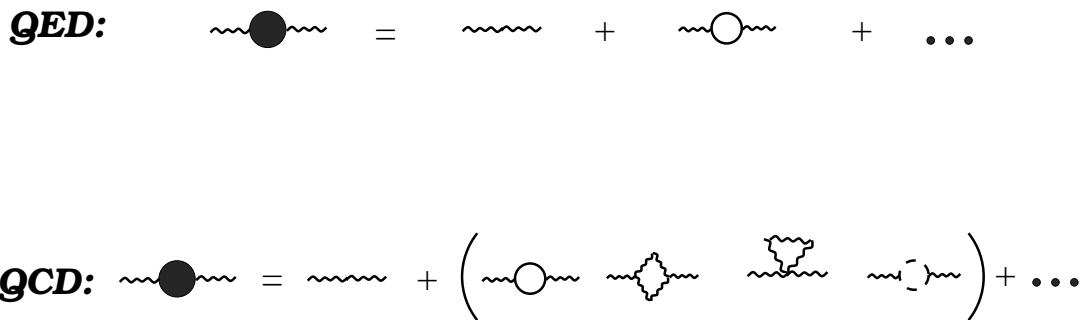

Figure 27: *(top) Photon and (bottom) gluon self-energy through one loop.*

We now compute the fermion loop graph in Fig. 28. As shown in Fig. 27, in the QED case the fermion loop is all that contributes to the self-energy, while in the QCD case this gives one of the required contributions.
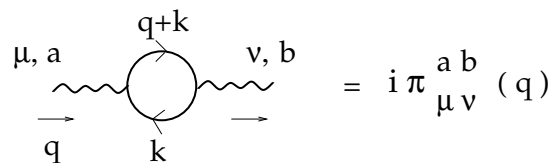

Figure 28: *Fermion loop contribution to the gauge boson self-energy.*

The graph in Fig. 28 is given by

$$i\pi_{\mu\nu}^{ab}(q) = -g^2 \, \text{Tr}(T^a T^b) \int \frac{d^4 k}{(2\pi)^4} \frac{\text{Tr}[\gamma_\mu(\not{k} + \not{q} + m)\gamma_\nu(\not{k} + m)]}{(k^2 - m^2 + i0^+)((k+q)^2 - m^2 + i0^+)} \quad . \tag{7.10}$$

This expression is written in general for the non-abelian case. In this case the color-charge factor is evaluated from Eqs. (6.21),(6.22) and equals

$$\text{Tr}(T^a T^b) = \frac{1}{2} \, \delta^{ab} \quad . \tag{7.11}$$

The QED case is obtained from Eq. (7.10) by taking

$$g^2 \longrightarrow e^2 \;\; = \;\; 4\pi\alpha \;\; ,$$
$$\text{Tr}(T^a T^b) \;\; \longrightarrow \;\; 1 \;\; . \tag{7.12}$$

The integral in Eq. (7.10) is ultraviolet divergent. By superficial power counting in the loop momentum k, the divergence is quadratic. Gauge invariance however requires that $\pi_{\mu\nu}$ be proportional to the transverse projector $g_{\mu\nu}q^2 - q_\mu q_\nu$, that is,

$$\pi_{\mu\nu} = \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) \Pi(q^2) \quad . \tag{7.13}$$

This reduces the degree of divergence by two powers of momentum. As a result, the divergence in Eq. (7.10) is not quadratic but logarithmic.

We need a regularization method to calculate the integral (7.10) and parameterize the divergence. We take the method of dimensional regularization. This consists of continuing the integral from 4 to $d = 4 - 2\varepsilon$ dimensions by introducing the dimensionful mass-scale parameter μ so that

$$g^2 \frac{d^4 k}{(2\pi)^4} \longrightarrow g^2(\mu^2)^\varepsilon \frac{d^{4-2\varepsilon} k}{(2\pi)^{4-2\varepsilon}} \quad . \tag{7.14}$$

In dimensional regularization a logarithmic divergence $d^4 k / k^4$ appears as a pole at $\varepsilon = 0$ (i.e., $d = 4$). We thus identify ultraviolet divergences in the integral (7.10) by identifying poles in $1/\varepsilon$.

By carrying out the calculation in dimensional regularization, the result for $\pi_{\mu\nu}$ is

$$\pi_{\mu\nu}^{ab}(q) \;\; = \;\; -\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) \text{Tr}(T^a T^b) \frac{g^2}{4\pi^2} \Gamma(\varepsilon) \int_0^1 dx \left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)^\varepsilon 2x(1-x)$$

$$\equiv \;\; \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) \Pi(q^2) \quad . \tag{7.15}$$

We can interpret the different factors in this result. As mentioned above, the first factor on the right hand side, consistent with the gauge-invariance requirement (7.13), implies that the gauge boson self-energy is purely transverse,

$$\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) q^\mu = \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) q^\nu = 0 \quad . \tag{7.16}$$

Owing to the transversality of the self-energy, loop corrections do not give mass to gauge bosons in QED and QCD. The factor $\text{Tr}(T^a T^b)$ in Eq. (7.15) is the non-abelian charge

factor, which just reduces to 1 in the QED case according to Eq. (7.12). Next, $g^2/(4\pi^2)$ is the coupling factor, which becomes $e^2/(4\pi^2) = \alpha/\pi$ in the QED case (7.12). The Euler gamma function $\Gamma(\varepsilon)$ contains the logarithmic divergence, i.e., the pole at $\varepsilon = 0$ ($d = 4$) in dimensional regularization:

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - C_E + \mathcal{O}(\varepsilon) \quad , \qquad C_E \simeq .5772 \ . \tag{7.17}$$

The first factor in the integrand of Eq. (7.15) results from the regularization method, depending on the ratio between the dimensional-regularization scale μ^2 and a linear combination of the physical mass scales m^2 and q^2. The last factor in the integrand, $2x(1-x)$, depends on the details of the calculated Feynman graph.

We can extract the ultraviolet divergent part of the self-energy by computing the integral in Eq. (7.15) at $q^2 = 0$. Higher q^2 powers in the expansion of $\Pi(q^2)$ give finite contributions. We have

$$\begin{aligned}
\Pi(0) &= -\text{Tr}(T^a T^b) \frac{g^2}{4\pi^2} \Gamma(\varepsilon) \int_0^1 dx \left(\frac{4\pi\mu^2}{m^2}\right)^\varepsilon 2x(1-x) \\
&\simeq -\text{Tr}(T^a T^b) \frac{g^2}{4\pi} \frac{1}{3\pi} \frac{1}{\varepsilon} + \ldots \ ,
\end{aligned} \tag{7.18}$$

where in the last line we have used the expansion (7.17) of the gamma function and computed the integral in dx. Specializing to the QED case according to Eq. (7.12) gives

$$\Pi(0) \simeq -\frac{\alpha}{3\pi} \frac{1}{\varepsilon} + \ldots \qquad \text{(QED)} \quad . \tag{7.19}$$

We will next use the results in Eqs. (7.15),(7.19) to discuss the renormalization of the electromagnetic coupling.

7.3   Renormalization of the electromagnetic coupling

Suppose we consider a physical process occurring via photon exchange, and ask what the effect is of the renormalization on the photon propagator. Fig. 29 illustrates this effect by multiple insertions of the photon self-energy,

$$D_0 \to D = D_0 + D_0 \pi D_0 + D_0 \pi D_0 \pi D_0 + \ldots \ , \tag{7.20}$$

where D_0 is the photon propagator given in Fig. 8 and π is the photon self-energy computed in Eq. (7.15). We can sum the series in Eq. (7.20) by applying repeatedly the transverse projector in π and using that longitudinal contributions vanish by gauge invariance, and we get

$$D_0 \to D = D_0 \frac{1}{1 + \Pi(q^2)} \ . \tag{7.21}$$

Then the effect of renormalization in the photon exchange process amounts to

$$\frac{e_0^2}{q^2} \longrightarrow \frac{e_0^2}{q^2} \frac{1}{1 - \Pi(q^2)} \ , \tag{7.22}$$
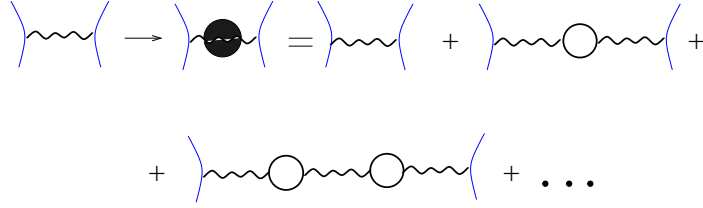
Figure 29: *Effect of renormalization in a photon exchange process.*

where q is the photon momentum.

Let us now rewrite the denominator on the right hand side of Eq. (7.22) by separating the divergent part and the finite part in Π. According to the discussion around Eq. (7.18), this can be achieved by

$$1 - \Pi(q^2) = [1 - \Pi(0)] \left[1 - \left(\Pi(q^2) - \Pi(0) \right) \right] + \mathcal{O}(\alpha^2) \ . \tag{7.23}$$

Therefore Eq. (7.22) gives

$$\frac{e_0^2}{q^2} \quad \longrightarrow \quad \frac{e_0^2}{q^2} \frac{1}{1 - \Pi(q^2)}$$

$$\simeq \quad \frac{1}{q^2} \underbrace{\frac{e_0^2}{1 - \Pi(0)}}_{e^2 \equiv Z_3 e_0^2} \underbrace{\frac{1}{1 - [\Pi(q^2) - \Pi(0)]}}_{q^2 - \text{dependence}} \ . \tag{7.24}$$

In the last line of Eq. (7.24) we have underlined two distinct effects in the result we obtain from renormalization. The first is that the strength of the coupling is modified to

$$\frac{e_0^2}{1 - \Pi(0)} \equiv e^2 \ , \tag{7.25}$$

from which, by comparison with Eq. (7.9), we identify the renormalization constant Z_3:

$$Z_3 \quad \simeq \quad 1 + \Pi(0)$$

$$= \quad 1 - \frac{\alpha}{3\pi} \frac{1}{\varepsilon} + \dots \ , \tag{7.26}$$

where in the last line we have used the explicit result for $\Pi(0)$ in Eq. (7.19). The coupling e in Eq. (7.25) is the physical coupling, that is, the renormalized coupling. This is obtained from the unrenormalized one, e_0, via a divergent, but unobservable, rescaling, according to the general procedure outlined below Eq. (7.3).

The second effect in Eq. (7.24) is that the coupling acquires a dependence on the momentum transfer q^2, controlled by the finite part of the self-energy, $\Pi(q^2) - \Pi(0)$. This dependence is free of divergences and observable. The q^2-dependence of the electromagnetic coupling is a new physical effect due to loop corrections. Using the explicit expression for Π in Eq. (7.15), we obtain that for low q^2

$$\Pi(q^2) - \Pi(0) \to 0 \qquad \text{for} \quad q^2 \to 0 \ , \tag{7.27}$$

and for high q^2

$$\Pi(q^2) - \Pi(0) \simeq \frac{\alpha}{3\pi} \ln \frac{q^2}{m^2} \qquad \text{for} \quad q^2 \gg m^2 \ . \tag{7.28}$$

Thus e^2 in Eq. (7.25) is the value of the coupling at $q^2 = 0$; the coupling increases as q^2 increases. Substituting Eqs. (7.25),(7.28) into Eq. (7.24) and rewriting it in terms of the fine structure, we have for large momenta

$$\alpha(q^2) = \frac{\alpha}{1 - (\alpha/(3\pi)) \ln(q^2/m^2)} \ . \tag{7.29}$$

The q^2-dependence of the coupling is referred to as running coupling. We will discuss this topic further in Sec. 8.

The result for the electromagnetic coupling that we have just found can be viewed as summing a series of perturbative large logarithms for $q^2 \gg m^2$. By expanding Eq. (7.29) in powers of α, we have

$$\begin{aligned} \alpha(q^2) &= \frac{\alpha}{1 - (\alpha/(3\pi)) \ln(q^2/m^2)} \\ &= \alpha \left(1 + \frac{\alpha}{3\pi} \ln \frac{q^2}{m^2} + \ldots + \frac{\alpha^n}{(3\pi)^n} \ln^n \frac{q^2}{m^2} + \ldots \right) \ . \end{aligned} \tag{7.30}$$

This is the simplest example of a conceptual framework referred to as resummation in QED and QCD. The point is that if the result (7.24) for the physical process is expressed in terms of an expansion in powers of α, as in Eq. (7.30), perturbative coefficients to higher orders are affected by large logarithmic corrections. On the other hand, one obtains a well-behaved perturbation series, without large higher-order coefficients, if the result is expressed in terms of the effective charge $\alpha(q^2)$.

7.4 Vertex correction and anomalous magnetic moment

In this section we study the one-loop vertex correction of Fig. 30. In particular we compute its contribution to the electron's magnetic moment,

$$\boldsymbol{\mu} = g \, \frac{e}{2m} \, \boldsymbol{S} \ , \tag{7.31}$$

where $\boldsymbol{S}$ is the spin operator and g is the gyromagnetic ratio. This computation gives

$$g = g_{\text{Dirac}} + \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2) \ , \quad g_{\text{Dirac}} = 2 \ , \tag{7.32}$$

where $g_{\text{Dirac}} = 2$ is the prediction from the Dirac equation and α/π is the correction from the graph in Fig. 30. Higher order corrections arise from multi-loop graphs. The deviations from the Dirac value are referred to as the electron's anomalous magnetic moment.

Let us consider first the Dirac equation coupled to electromagnetism, Eq. (3.20), and write the magnetic interaction term explicitly. We can recast Eq. (3.20) in the two-component notation of Subsec. 2.4, including the electromagnetic coupling, as

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \boldsymbol{\sigma} \cdot (\mathbf{p} - e\boldsymbol{A}) \\ \boldsymbol{\sigma} \cdot (\mathbf{p} - e\boldsymbol{A}) & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} \ . \tag{7.33}$$
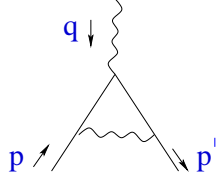
Figure 30: *One-loop vertex correction in QED.*

Now substitute the bottom equation in (7.33) into the top equation, use the Pauli σ matrix relation

$$\boldsymbol{\sigma} \cdot \mathbf{a} \, \boldsymbol{\sigma} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i \, \boldsymbol{\sigma} \cdot \mathbf{a} \wedge \mathbf{b} \ , \tag{7.34}$$

and take the nonrelativistic limit $E \simeq m$, in which $\phi \ll \chi$. We then obtain that the action of the hamiltonian on the spinor ψ can be written as

$$H\psi \simeq \left((\mathbf{p} - e\boldsymbol{A})^2 - \frac{e}{2m} \, \mathbf{B} \cdot 2\boldsymbol{S} \right) \psi \ , \tag{7.35}$$

where $\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}$ is the magnetic field and $\boldsymbol{S}$ is the spin operator given in terms of the σ matrices in Eq. (2.57). We recognize that the second term in the right hand side of Eq. (7.35) is the magnetic interaction

$$-\boldsymbol{\mu} \cdot \mathbf{B} \ , \quad \text{with} \ \ \boldsymbol{\mu} = \frac{e}{2m} \, 2\boldsymbol{S} \ . \tag{7.36}$$

That is, the Dirac equation prediction for the gyromagnetic ratio g in Eq. (7.31) is

$$g_{\text{Dirac}} = 2 \ . \tag{7.37}$$

Let us consider now the vertex function $\Gamma^\nu(p, p')$ represented at one loop in Fig. 30. We can determine the general structure of the vertex function based on relativistic invariance and gauge invariance. Because $\Gamma^\nu(p, p')$ transforms like a Lorentz vector, we can write it as a linear combination of γ^ν, p^ν, p'^ν, or equivalently

$$\Gamma^\nu(p, p') = A \, \gamma^\nu + B(p + p')^\nu + C(p - p')^\nu \ , \tag{7.38}$$

where A, B and C are scalar functions of q^2 only ($q = p' - p$).

Gauge invariance requires

$$q_\nu \Gamma^\nu = 0 \ . \tag{7.39}$$

By dotting q_ν into Eq. (7.38), the term in B gives zero, and the term in A gives zero once it is sandwiched between $\bar{u}(p')$ and $u(p)$. Thus $C = 0$. We can further show that the following identity holds,

$$\bar{u}(p')\gamma^\nu u(p) = \frac{1}{2m} \, \bar{u}(p')(p + p')^\nu u(p) + \frac{i}{m} \, \bar{u}(p') \, \Sigma^{\nu\rho} \, q_\rho \, u(p) \ , \tag{7.40}$$

where Σ is given in Eq. (2.28), $\Sigma^{\nu\rho} \equiv (i/4) \, [\gamma^\nu, \gamma^\rho]$. This implies that the term in $(p+p')^\nu$ in Eq. (7.38) can be traded for a linear combination of a term in γ^ν and a term in $\Sigma^{\nu\rho} q_\rho$. Therefore the vertex function can be decomposed in general as

$$\Gamma^\nu(p, p') = F_1(q^2) \, \gamma^\nu + \frac{i}{m} \, F_2(q^2) \, \Sigma^{\nu\rho} \, q_\rho \ , \tag{7.41}$$

where the scalar functions $F_1(q^2)$ and $F_2(q^2)$ are the electron's electric and magnetic form factors. At tree level, $\Gamma^\nu = \gamma^\nu$, thus $F_1 = 1$ and $F_2 = 0$. In general, F_1 and F_2 receive radiative corrections from loop graphs and are related to the electron's charge and magnetic moment,

$$F_1(0) = Q \quad , \tag{7.42}$$

$$F_2(0) = \frac{g-2}{2} \quad , \tag{7.43}$$

where Q is the electron's charge in units of e and g is the electron's magnetic moment in units of $(e/(2m))S$ where S is the electron spin. $F_1(0)$ is 1 to all orders, that is, radiative corrections to F_1 vanish at $q^2 = 0$. We next compute the correction to $F_2(0)$ at one loop.

To this end, consider the one-loop graph in Fig. 30. This is given by

$$\bar{u}(p')ie\Gamma^\nu u(p) = e^3 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p') \gamma^\lambda (\slashed{k} + \slashed{q} + m) \gamma^\nu (\slashed{k} + m) \gamma_\lambda u(p)}{[(k+q)^2 - m^2 + i\varepsilon] [k^2 - m^2 + i\varepsilon] [(p-k)^2 + i\varepsilon]} \ . \tag{7.44}$$

The integral in Eq. (7.44) can be handled starting with the following Feynman's parameterization of the three denominators in the integrand,

$$\frac{1}{[(k+q)^2 - m^2 + i\varepsilon] [k^2 - m^2 + i\varepsilon] [(p-k)^2 + i\varepsilon]}$$

$$= \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \frac{2\,\delta(x_1 + x_2 + x_3 - 1)}{[x_1\,((k+q)^2 - m^2) + x_2\,(k^2 - m^2) + x_3\,(p-k)^2 + i\varepsilon]^3}$$

$$= \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \frac{2\,\delta(x_1 + x_2 + x_3 - 1)}{(\widetilde{k}^2 - K + i\varepsilon)^3} \quad , \tag{7.45}$$

where in the last line we have set $\widetilde{k} = k + x_1 q - x_3 p$, $K = m^2(1 - x_3)^2 - q^2 x_1 x_2$.

Next change integration variable $k \to \widetilde{k}$ in Eq. (7.44), and note that the numerator in the integrand can be rewritten according to

$$\gamma^\lambda (\slashed{k} + \slashed{q} + m) \gamma^\nu (\slashed{k} + m) \gamma_\lambda \tag{7.46}$$

$$= \gamma^\nu [\widetilde{k}^2 - 2q^2(1 - x_1)(1 - x_2) + 2m^2(4x_3 - 1 - x_3^2)] - 4mi\Sigma^{\nu\rho}q_\rho x_3(1 - x_3) \ .$$

Then Eq. (7.44) can be recast in the form

$$\bar{u}(p')\Gamma^\nu u(p) = -ie^2 \bar{u}(p') \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\, 2\,\delta(x_1 + x_2 + x_3 - 1)$$

$$\times \int \frac{d^4\widetilde{k}}{(2\pi)^4} \left[\gamma^\nu \frac{\widetilde{k}^2 - 2q^2(1 - x_1)(1 - x_2) - 2m^2(1 - 4x_3 + x_3^2)}{[\widetilde{k}^2 - K]^3} \right.$$

$$+ \left. \frac{i}{m} \Sigma^{\nu\rho}q_\rho \frac{-4m^2 x_3(1 - x_3)}{[\widetilde{k}^2 - K]^3} \right] u(p) \quad . \tag{7.47}$$

Comparing Eq. (7.47) with the general decomposition in Eq. (7.41), we see that the two terms in the second and third line of Eq. (7.47) give one-loop integral representations for,

respectively, the form factors $F_1(q^2)$ and $F_2(q^2)$. Let us concentrate on the calculation of F_2:

$$
\begin{aligned}
F_2(q^2) &= -ie^2 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \; 2 \; \delta(x_1 + x_2 + x_3 - 1) \\
&\times \int \frac{d^4\widetilde{k}}{(2\pi)^4} \frac{-4m^2 x_3(1-x_3)}{[\widetilde{k}^2 - K]^3} \quad .
\end{aligned}
\tag{7.48}
$$

While the integral for F_1 in Eq. (7.47) has divergences that need regularization, the integral (7.48) for F_2 is finite. Let us compute the result for $q^2 = 0$.

The integration over the four-momentum $\widetilde{k}$ in Eq. (7.48) can be done by using the transformation of variables $\widetilde{k}^0 \to -e^{i\pi/2}\widetilde{k}^0$ in the integral over the time component of the momentum. This yields the result

$$
\int \frac{d^4\widetilde{k}}{(2\pi)^4} \frac{1}{[\widetilde{k}^2 - K]^3} = -\frac{i}{32\pi^2 K} \quad .
\tag{7.49}
$$

Then we have $(e^2 = 4\pi\alpha)$

$$
\begin{aligned}
F_2(0) &= \frac{\alpha}{\pi} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \; \delta(x_1 + x_2 + x_3 - 1) \frac{m^2 x_3(1-x_3)}{(1-x_3)^2 m^2} \\
&= \frac{\alpha}{\pi} \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \frac{x_3}{1-x_3} \\
&= \frac{\alpha}{\pi} \int_0^1 dx_3(1-x_3) \frac{x_3}{1-x_3} = \frac{\alpha}{2\pi} \quad .
\end{aligned}
\tag{7.50}
$$

We thus obtain that the one-loop contribution to the electron's anomalous magnetic moment $g - 2 = 2F_2(0)$ is given by

$$
g - 2 = 2F_2(0) = \frac{\alpha}{\pi} \quad .
\tag{7.51}
$$

8 Renormalization group

Let us discuss renormalization from the standpoint of the renormalization group. We have seen that renormalization introduces dependence on a renormalization scale μ in loop calculations. As the value of μ is arbitrary, physics must be invariant under changes in this scale. This invariance is expressed in a precise manner by the renormalization group. We will see that by studying the dependence on the renormalization scale μ we gain insight into the asymptotic behavior of the theory at short distances.

8.1 Renormalization scale dependence and evolution equations

In this section we illustrate how the relation between renormalized and unrenormalized quantities, applied to a given physical quantity G, can be used to to study the dependence on the renormalization scale μ and to obtain renormalization group evolution equations.

Renormalizability implies that the divergent dependence in the unrenormalized quantity G_0 can be factored out in the renormalization constant Z, provided we re-express renormalized G in terms of the renormalized coupling and renormalization scale μ,

$$G_0(p_i, \alpha_0) = Z G(p_i, \alpha, \mu) \ . \tag{8.1}$$

Here p_i is the set of physical momenta on which G depends, α is the renormalized coupling and α_0 is the unrenormalized coupling. Because the left hand side in Eq. (8.1) does not depend on μ,

$$\frac{d}{d \ln \mu^2} \, G_0 = 0 \ , \tag{8.2}$$

we have

$$\frac{d}{d \ln \mu^2} \, (ZG) = 0 \quad \Longrightarrow \quad \frac{\partial G}{\partial \ln \mu^2} + \frac{\partial G}{\partial \alpha} \frac{\partial \alpha}{\partial \ln \mu^2} + \frac{\partial \ln Z}{\partial \ln \mu^2} \, G = 0 \ . \tag{8.3}$$

By defining

$$\beta(\alpha) = \frac{\partial \alpha}{\partial \ln \mu^2} \ , \tag{8.4}$$

$$\gamma(\alpha) = \frac{\partial \ln Z}{\partial \ln \mu^2} \ , \tag{8.5}$$

we can rewrite Eq. (8.3) as

$$\left[\frac{\partial}{\partial \ln \mu^2} + \beta(\alpha) \, \frac{\partial}{\partial \alpha} + \gamma(\alpha) \right] G(p_i, \alpha, \mu) = 0 \ , \tag{8.6}$$

where $\beta(\alpha)$ and $\gamma(\alpha)$ are calculable functions of α.

Suppose we measure G at a physical mass-scale Q. Let us rescale by Q the arguments in G and set

$$G(p_i, \alpha, \mu) = F(x_i, t, \alpha) \ , \tag{8.7}$$

where

$$x_i = \frac{p_i}{Q} \ , \qquad t = \ln \frac{Q^2}{\mu^2} \ . \tag{8.8}$$

In this notation Eq. (8.6) can be written as

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha)\,\frac{\partial}{\partial \alpha} + \gamma(\alpha) \right] F(t, \alpha) = 0 \ , \tag{8.9}$$

where from now on we will not write explicitly the dependence on the rescaled physical momenta x_i in F.

Eq. (8.9) is the renormalization group evolution equation, which we can solve with boundary condition $F(0, \alpha)$ at $t = 0$, i.e., $\mu = Q$. To do this, we first write the solution for the case $\gamma = 0$ and then generalize this solution to any γ.

For $\gamma = 0$ we have

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha)\,\frac{\partial}{\partial \alpha} \right] F(t, \alpha) = 0 \qquad (\gamma = 0) \quad . \tag{8.10}$$

Now observe that if we construct $\alpha(t)$ such that

$$t = \int_{\alpha}^{\alpha(t)} \frac{d\alpha'}{\beta(\alpha')} \ , \tag{8.11}$$

then any F of the form

$$F(t, \alpha) = F(0, \alpha(t)) \tag{8.12}$$

satisfies the equation and the boundary condition.

Eq. (8.11) defines $\alpha(t)$ as an implicit function. To verify that Eq. (8.12) is solution, note first that the boundary condition at $t = 0$ is

$$t = 0 \ , \quad \alpha(0) = \alpha \quad \Longrightarrow \quad F = F(0, \alpha) \ . \tag{8.13}$$

Next evaluate the derivative of Eq. (8.11) with respect to t,

$$1 = \frac{1}{\beta(\alpha(t))}\,\frac{\partial \alpha(t)}{\partial t} \ , \tag{8.14}$$

and with respect to α,

$$0 = \frac{1}{\beta(\alpha(t))}\,\frac{\partial \alpha(t)}{\partial \alpha} - \frac{1}{\beta(\alpha)} \ . \tag{8.15}$$

Then the differential operator in Eq. (8.10) applied to $F(0, \alpha(t))$ gives

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha)\,\frac{\partial}{\partial \alpha} \right] F(0, \alpha(t))$$

$$= -\frac{\partial F}{\partial \alpha(t)} \left[\underbrace{\frac{\partial \alpha(t)}{\partial t}}_{\beta(\alpha(t))} - \beta(\alpha)\,\underbrace{\frac{\partial \alpha(t)}{\partial \alpha}}_{\beta(\alpha(t))/\beta(\alpha)} \right] = 0 \ , \tag{8.16}$$

where in the last line we have used Eqs. (8.14),(8.15).

In the general case $\gamma \neq 0$, the solution to Eq. (8.9) is obtained from the $\gamma = 0$ answer (8.12) by multiplication by the exponential of a γ integral, as follows

$$
\begin{aligned}
F(t, \alpha) &= F(0, \alpha(t)) \exp\left[\int_\alpha^{\alpha(t)} d\alpha' \, \frac{\gamma(\alpha')}{\beta(\alpha')}\right] \\
&= F(0, \alpha(t)) \exp\left[\int_0^t dt' \, \gamma(\alpha(t'))\right] \quad .
\end{aligned}
\tag{8.17}
$$

In the second line in Eq. (8.17) we have made the integration variable transformation using Eq. (8.11). We can verify that Eq. (8.17) is solution by a method similar to that employed above for the case $\gamma = 0$.

Eq. (8.17) indicates that once ultraviolet divergences are removed through renormalization, all effects of varying the scale in F from μ to Q can be taken into account by i) replacing α by $\alpha(t)$, and ii) including the t-dependence given by the exponential factor in γ. The latter factor breaks scaling in t, modifying the "engineering" dimensions of F by γ-dependent terms. For this reason γ is referred to as anomalous dimension. By expanding the exponential factor in powers of the coupling, we see that this factor sums terms of the type $(\alpha t)^n$ to all orders in perturbation theory. Eq. (8.17) thus provides a second example, besides that seen in Eq. (7.30) for the electric charge, of perturbative resummation of logarithmic corrections to all orders in the coupling, giving rise to an improved perturbation expansion, in which coefficients of higher order are free of large logarithms.

In QCD the $e^+ e^-$ annihilation cross section $\sigma(e^+ e^- \to \text{hadrons})$ is an example of the $\gamma = 0$ case in Eq. (8.12), while deep-inelastic scattering structure functions are an example of the $\gamma \neq 0$ case in Eq. (8.17).

8.2 RG interpretation of the photon self-energy

Let us revisit the analysis of the photon self-energy in Sec. 7 from the standpoint of the renormalization group. The divergent part of the renormalization constant Z_3 computed in Eq. (7.26) determines the QED β function at one loop.

According to Eq. (8.4), the variation of the coupling α with the energy scale μ is governed by the β function, calculable as a function of α. In dimensional regularization, from

$$
\alpha \left(\mu^2\right)^\varepsilon = Z_3 \, \alpha_0 \quad ,
\tag{8.18}
$$

by using Eq. (7.26) we have

$$
\begin{aligned}
\frac{\partial \alpha}{\partial \ln \mu^2} &= -\varepsilon \left(1 - \frac{\alpha}{3\pi} \frac{1}{\varepsilon}\right) \alpha_0 \left(\mu^2\right)^{-\varepsilon} \\
&= \frac{1}{3\pi} \, \alpha^2 \quad .
\end{aligned}
\tag{8.19}
$$

The leading term of the QED β function at small coupling is given by (Fig. 31),

$$
\begin{aligned}
\beta(\alpha) &= b\alpha^2 + \mathcal{O}(\alpha^3) \quad , \\
b &= \frac{1}{3\pi} \quad .
\end{aligned}
\tag{8.20}
$$

Inserting the result (8.20) into Eq. (8.4) gives the differential equation

$$\frac{\partial \alpha}{\partial \ln \mu^2} = b\alpha^2 \quad . \tag{8.21}$$

This can be solved by

$$\frac{d\alpha}{\alpha^2} = b \, \frac{d\mu^2}{\mu^2} \quad \Longrightarrow \quad -\frac{1}{\alpha(q^2)} + \frac{1}{\alpha} = b \ln \frac{q^2}{q_0^2} \quad , \tag{8.22}$$

which gives

$$\alpha(q^2) = \frac{\alpha}{1 - b\alpha \ln(q^2/q_0^2)} \quad , \qquad b = 1/(3\pi) \quad , \tag{8.23}$$

that is, the result (7.29) derived directly in Subsec. 7.3.
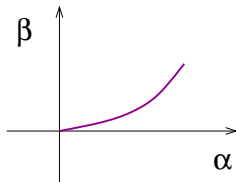
Figure 31: *Small-coupling approximation to the β function in QED.*

To sum up, we have found from the analysis of electric charge renormalization in Sec. 7.3 and in this section that as a result of loop graphs the electromagnetic coupling is energy-dependent. We can regard this result as illustrating the breaking of scale invariance as an effect of the quantum corrections taken into account by renormalization. We start at tree level with a coupling that is scale invariant. Then we include loops. This implies introducing an unphysical mass scale, such as the renormalization scale μ, to treat quantum fluctuations at short distances, or high momenta. At the end of the calculation in the renormalized theory, the unphysical mass scale disappears from physical quantities. But an observable, physical effect from including loop corrections remains in the fact that scale invariance is broken. The physical coupling depends on the energy scale at which we probe the interaction. The renormalization group provides the appropriate framework to describe this phenomenon, in which the rescalings (7.4) of the couplings and wave functions, necessary to compensate variations in the arbitrary renormalization scale, are governed by universal functions, respectively the β and γ functions (8.4),(8.5) of the theory.

8.3   QCD β function at one loop

We now extend the discussion to the case of renormalization in QCD at one loop, and determine the one-loop β function.

In the QCD case we assign rescaling relations analogous to those in Eq. (7.4) for the abelian theory. For wave function and mass renormalization we set

$$A \to A_0 \;\; = \;\; \sqrt{Z_3} \, A \;\; ,$$

$$\psi \to \psi_0 = \sqrt{Z_2}\,\psi \;,$$

$$c \to c_0 = \sqrt{\widetilde{Z}_3}\,c \;,$$

$$m \to m_0 = \frac{Z_m}{Z_2}\,m \;. \tag{8.24}$$

where, in addition to the renormalization constants of the abelian case, we introduce $\widetilde{Z}_3$ for ghost renormalization. For renormalization of quark-gluon, ghost-gluon and gluon self-coupling vertices we set

$$Z_2\sqrt{Z_3}g_0 = Z_1\,g \;,$$

$$\widetilde{Z}_3\sqrt{Z_3}g_0 = \widetilde{Z}_1\,g \;,$$

$$Z_3^{3/2}g_0 = Z_{1,3}\,g \;,$$

$$Z_3^2 g_0^2 = Z_{1,4}\,g \;. \tag{8.25}$$

As noted in Sec. 6.2, non-abelian gauge invariance requires that the vertices have equal couplings. This implies relations among the different Z in Eq. (8.25), as follows

$$\frac{\widetilde{Z}_1}{\widetilde{Z}_3} = \frac{Z_1}{Z_2} = \frac{Z_{1,3}}{Z_3} = \sqrt{\frac{Z_{1,4}}{Z_3}} \;. \tag{8.26}$$

In the non-abelian theory, unlike QED, in general one has $Z_1 \neq Z_2$. The relations in Eq. (8.26) can be seen as non-abelian generalizations of the QED result $Z_1 = Z_2$ given in Eq. (7.8).

We can define the renormalized coupling from the quark-gluon vertex. The analogue of Eq. (8.18) for the QCD case is

$$\alpha_s \left(\mu^2\right)^\varepsilon = \frac{Z_2^2}{Z_1^2}\,Z_3\,\alpha_{s0} \;. \tag{8.27}$$

Each of the renormalization constants Z_i has a perturbation series expansion, with the coefficients of the expansion being ultraviolet divergent. In dimensional regularization the ultraviolet divergences appear as poles at $\varepsilon = 0$, so that the Z_i have the form

$$Z_i = 1 + \alpha_s\,\frac{1}{\varepsilon}\,c_i + \text{ finite } \;, \tag{8.28}$$

where the coefficients c_i of the divergent terms are to be calculated. By using Eqs. (8.27) and (8.28), the β function is given by

$$\begin{aligned}
\beta(\alpha_s) &= \frac{\partial \alpha_s}{\partial \ln \mu^2} \\
&= -\varepsilon \alpha_B \left(\mu^2\right)^{-\varepsilon} [1 - 2(Z_1 - 1) + 2(Z_2 - 1) + (Z_3 - 1)] \\
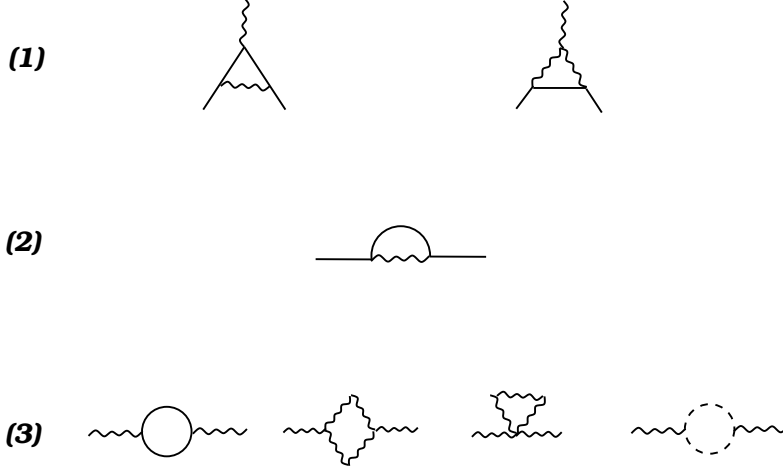&= 2\alpha_s^2 (c_1 - c_2 - \frac{1}{2}\,c_3) \;. \tag{8.29}
\end{aligned}$$

Figure 32: *One-loop corrections to (1) quark-gluon vertex; (2) quark self-energy; (3) gluon self-energy.*

The Feynman graphs contributing to c_1, c_2 and c_3 are the one-loop graphs for, respectively, the quark-gluon vertex renormalization, quark self-energy renormalization, and gluon self-energy renormalization, and they are shown in Fig. 32. The calculation of these graphs proceeds similarly to the calculation done in Sec. 7.2 for the fermion loop contribution. By computing these graphs, working in Feynman gauge $\xi = 1$ (as in Fig. 19), we obtain the results for the renormalization constants Z_i,

$$Z_1 = 1 - \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \left(C_F + C_A \right) \ , \tag{8.30}$$

$$Z_2 = 1 - \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} C_F \ , \tag{8.31}$$

$$Z_3 = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \left(\frac{5}{3} C_A - \frac{4}{3} N_f T_F \right) \ , \tag{8.32}$$

where N_f is the number of quark flavors (Sec. 6.1), and the color charge factors are given in Sec. 6.3,

$$C_A = N = 3 \ , \quad C_F = \frac{N^2 - 1}{2N} = \frac{4}{3} \ , \quad T_F = \frac{1}{2} \ . \tag{8.33}$$

Note from the expression for Z_3 that the second term in the bracket in Eq. (8.32) is the term computed in Sec. 7.2 from the fermion loop graph, which, in the abelian limit $N_f T_F \to 1$, gives the QED contribution $-\alpha/(3\pi\varepsilon)$ of Eq. (7.26).

From Eqs. (8.30)-(8.32) we read the coefficients c_i to be put into Eq. (8.29) to determine the β function. We obtain

$$
\begin{aligned}
\beta(\alpha_s) &= 2\alpha_s^2 (c_1 - c_2 - \frac{1}{2} c_3) = 2 \frac{\alpha_s^2}{4\pi} \left(-C_F - C_A + C_F - \frac{1}{2} \frac{5}{3} C_A + \frac{1}{2} \frac{4}{3} N_f T_R \right) \\
&= \frac{\alpha_s^2}{4\pi} \left(-\frac{11}{3} C_A + \frac{4}{3} N_f T_R \right) = -\frac{\alpha_s^2}{12\pi} \left(11N - 2N_f \right) \ .
\end{aligned}
\tag{8.34}
$$

Eq. (8.34) shows that for $N_f < 11N/2$ the β function in the non-abelian case has negative sign at small coupling (Fig. 33),

$$\beta(\alpha_s) = -\beta_0 \alpha_s^2 + \mathcal{O}(\alpha_s^3) \ , \tag{8.35}$$

where

$$\beta_0 = \frac{1}{12\pi}(11N - 2N_f) \ . \tag{8.36}$$

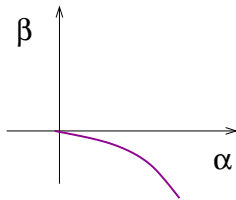This behavior of the β function is opposite to the behavior of the β function in QED, Eq. (8.20) (Fig. 31).

Figure 33: *Small-coupling approximation to the β function in QCD.*

The behavior of the β function in Eqs. (8.35),(8.36) implies that QCD is asymptotically free, i.e., weakly coupled at short distances. By inserting Eq. (8.35) into the renormalization group evolution equation,

$$\frac{\partial \alpha_s}{\partial \ln \mu^2} = \beta(\alpha_s) \simeq -\beta_0 \alpha_s^2 \ , \tag{8.37}$$

and solving Eq. (8.37), we obtain

$$\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \ \alpha_s(\mu^2) \ln q^2/\mu^2} \ , \tag{8.38}$$

where β_0 is given in Eq. (8.36). Eq. (8.38) expresses the q^2-dependence of the QCD running coupling at one loop. The QCD coupling decreases logarithmically as the momentum scale q^2 increases. This property is the basis for the perturbative calculability of scattering processes due to strong interactions at large momentum transfers.

8.4 The QCD scale Λ

From Eq. (8.38) we also see that QCD becomes strongly coupled in the infrared, low-momentum region. This behavior is opposite to that in QED. In the QED case, taking $q_0 \sim m$ in Eq. (8.23), with m the electron mass, we have strong coupling in the ultraviolet region for

$$q^2 \sim m^2 e^{3\pi/\alpha} \ , \tag{8.39}$$

corresponding to enormously high energies.

In the QCD case, calling Λ the mass scale at which the denominator in Eq. (8.38) vanishes, we have

$$1 + \beta_0 \ \alpha_s(\mu^2) \ln \frac{\Lambda^2}{\mu^2} = 0 \quad \Longrightarrow \quad \Lambda^2 = \mu^2 e^{-1/(\beta_0 \alpha_s(\mu^2))} \ . \tag{8.40}$$

In QED and QCD we thus get the different pictures in Fig. 34 for the scale, referred to as the Landau pole, at which the coupling becomes strong.
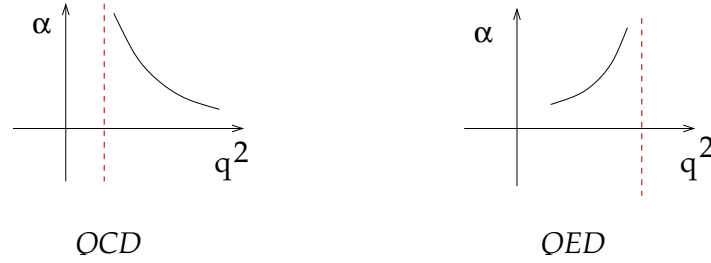

Figure 34: *Landau pole pictures in QCD and in QED.*

The scale Λ in Eq. (8.40) is renormalization-group invariant, i.e., it is independent of μ. Under transformations

$$\mu^2 \longrightarrow \mu'^2 = \mu^2 e^t \ ,$$

$$\alpha_s(\mu^2) \longrightarrow \alpha_s(\mu'^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \alpha_s(\mu^2)t} \ , \tag{8.41}$$

we have

$$\Lambda^2 \longrightarrow \mu'^2 e^{-1/(\beta_0 \alpha_s(\mu'^2))} \ ,$$

$$= \mu^2 e^t e^{-(1+\beta_0 \alpha_s(\mu^2)t)/(\beta_0 \alpha_s(\mu^2))} = \mu^2 e^t e^{-1/(\beta_0 \alpha_s(\mu^2))} e^{-t} = \Lambda^2 \ . \tag{8.42}$$

The scale Λ is a physical mass scale of the theory of strong interaction. Its measured value is about 200 MeV.
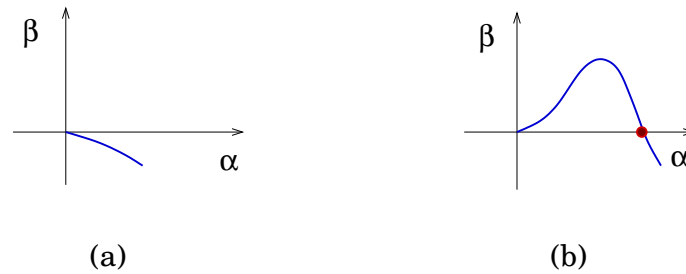

(a)   (b)

Figure 35: *(a) Trivial and (b) nontrivial ultraviolet fixed points of the β function.*

The running coupling (8.38) can be equivalently expressed in terms of Λ,

$$\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \, \alpha_s(\mu^2) \left[\ln(q^2/\Lambda^2) - 1/(\beta_0 \, \alpha_s(\mu^2))\right]}$$

$$= \frac{1}{\beta_0 \ln(q^2/\Lambda^2)} \ . \tag{8.43}$$

The rewriting (8.43) of Eq. (8.38) makes it manifest that the running coupling α_s does not depend on the choice of the renormalization scale μ.

Remark. The zero of the QCD β function at the origin, sketched in Fig. 35a, is responsible for the theory being weakly coupled at short distances. This behavior is referred to as a trivial ultraviolet fixed point. A behavior such as that in Fig. 35b (nontrivial ultraviolet fixed point), leading to strong coupling at short distances, is in principle possible but not realized in nature as far as we know. This is the reason why renormalization can be understood perturbatively and Feynman graphs provide a very effective method to investigate physical theories of fundamental interactions.

Acknowledgments

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