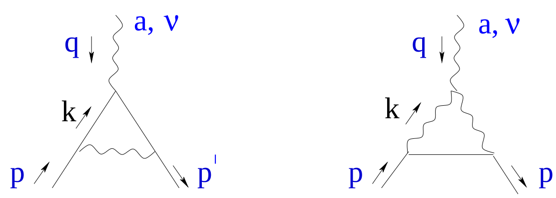
Nonabelian Quantum Field Theory

PROBLEM SET B (due March 9)

Problem 1

Consider the nonabelian fermion-fermion-gauge 3-point function at one loop:



(a) (b)

i) Work in dimensional regularization with $d = 4 - 2\varepsilon$ space-time dimensions. Show that the ultraviolet-divergent parts of graphs (a) and (b) are given in Feynman gauge by

$$\text{graph (a)} = \frac{ig^3}{(4\pi)^2}(C_F - \frac{1}{2}C_A)t^a\gamma^\nu\frac{1}{\varepsilon} + \ldots$$

$$\text{graph (b)} = \frac{ig^3}{(4\pi)^2}\frac{3}{2}C_A t^a\gamma^\nu\frac{1}{\varepsilon} + \ldots$$

ii) Determine the renormalization constant Z_1.

Problem 2

Specialize the 3-point function of Problem 1 to the U(1) gauge symmetry case:

$$\bar{u}(p')\ ie\Gamma^\nu u(p) = e^3 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\ \gamma^\lambda\ (\not{k} + \not{q} + m)\ \gamma^\nu\ (\not{k} + m)\ \gamma_\lambda\ u(p)}{[(k+q)^2 - m^2 + i\varepsilon]\ [k^2 - m^2 + i\varepsilon]\ [(p-k)^2 + i\varepsilon]} \quad .$$

(a) Using relativistic invariance and gauge invariance, show that the vertex function Γ^ν can be decomposed as follows:

$$\Gamma^\nu(p,p') = F_1(q^2)\ \gamma^\nu + \frac{i}{2m}\ F_2(q^2)\ \sigma^{\nu\rho}\ q_\rho\ ,$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$ and $q^\mu = p'^\mu - p^\mu$. The functions $F_1(q^2)$ and $F_2(q^2)$ are scalar functions representing the fermion's electric and magnetic form factors.

(b) Obtain a one-loop integral representation for $F_1(q^2)$ and for $F_2(q^2)$. Show that F_2 has neither ultraviolet nor infrared divergences.
(c) Calculate $F_2(0)$, and obtain that the one-loop correction to the electron's magnetic moment $g-2 \equiv 2F_2(0)$ is given by

$$g - 2 = \frac{\alpha}{\pi} \ .$$

Problem 3

Consider the cross section σ for $e^+e^- \to$ *hadrons* as a function of the total momentum square Q^2, the renormalization scale μ and the coupling α_s at scale μ:

$$\begin{aligned} \sigma(Q^2, \alpha_s(\mu), \mu/Q) &= \sigma_0(Q^2) \ \Big\{1 + [c_1 + c_1' \ln(\mu^2/Q^2)]\alpha_s(\mu) \\ &+ [c_2 + c_2' \ln(\mu^2/Q^2) + c_2'' \ln^2(\mu^2/Q^2)]\alpha_s^2(\mu) + \mathcal{O}(\alpha_s^3)\Big\} \ . \end{aligned}$$

Here the c's are perturbatively-calculable numerical coefficients.

i) Show that the lowest-order contribution σ_0 from the electromagnetic coupling of N_f species of quarks is given by

$$\sigma_0(Q^2) = \frac{4 \ \pi \ \alpha^2}{3 \ Q^2} \ N_c \sum_{f=1}^{N_f} e_f^2 \ ,$$

where $N_c = 3$, and e_f are the quark electric charges. [Work at Q^2 much larger than any of the fermion masses, and set these to zero.]

ii) Discuss the conditions on the higher-order coefficients imposed by renormalization group invariance of the cross section. Determine the value of c_2''. Determine the relation between c_2' and c_1. [Consider a renormalization group transformation $\mu \to Q$, $\alpha_s(\mu) \to \alpha_s(Q)$. Use the beta function to relate the values of the coupling at mass scales μ and Q, order-by-order.]

Problem 4

Consider the radiative correction from one-gluon emission in e^+e^- annihilation to hadrons:

i) Show that the final-state distribution in momentum fractions $x_i = (2p_i \cdot q)/q^2$ is given by

$$\frac{1}{\sigma_0} \frac{d\sigma}{dx_1 dx_2} = \frac{\alpha_s}{2\pi} \ C_F \ \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \ .$$

This is the leading contribution to the cross section for three-jet production.

ii) Compute the order-α_s correction to the total cross section σ by combining the above contribution with the one-loop $q\bar{q}$ contribution. Verify the cancellation of the infrared divergence, and show that

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma_0 \left[1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right] \quad .$$

Problem 5

Consider the axial currents of QCD with two species of quarks,

$$j_A^{\mu k} = \bar{\psi}\gamma^\mu\gamma^5\frac{\sigma^k}{2}\psi \ , \quad j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \ , \quad \psi = \begin{pmatrix} u \\ d \end{pmatrix} \ .$$

a) Write the chiral anomaly group-theory factors, and show that the isosinglet current j_A^μ has an anomaly from QCD interactions while the isotriplet current $j_A^{\mu k}$ does not. b) By the same method show that the current $j_A^{\mu k}$ has an anomaly from quarks' electromagnetic interactions.

Problem 6

A linear sigma model with $N = 2$ is coupled to a massless Dirac field via the interaction

$$\mathcal{L}_I = -g\overline{\psi}(\phi_1 + i\gamma^5\phi_2)\psi \quad ,$$

where ϕ_1 and ϕ_2 are real scalar fields and ψ is the fermion field.

(a) Take global $O(2)$ and chiral symmetry transformations of the fields,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad , \quad \psi \to e^{i\theta\gamma^5}\psi \quad ,$$

and determine the relation between α and θ such that the theory is globally invariant.

(b) Now let the scalar field doublet acquire nonzero vacuum expectation value,

$$\langle\phi\rangle = \begin{pmatrix} v \\ 0 \end{pmatrix} \ .$$

Show that the global symmetry of (a) is spontaneously broken; show that the fermion field ψ acquires a mass, and determine its expression.

Problem 7

The abelian Higgs model can be applied in the nonrelativistic case to the electrodynamics of a superconductor, interpreting $|\phi|^2$ as the density of Cooper pairs of electrons. Take

$$\mathcal{L} = D_\mu\phi^\dagger D^\mu\phi - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad ,$$

$$\text{where} \quad D_\mu = \partial_\mu - ieA_\mu \quad , \quad V(\phi) = -\mu^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 \quad .$$

Work in $A_0 = 0$ gauge, and consider the stationary case. i) Verify that the spontaneous breaking of the gauge symmetry by $\langle\phi\rangle = \sqrt{2\mu^2/\lambda} = v/\sqrt{2}$ induces the current

$$\mathbf{J} = e^2 v^2 \mathbf{A} \quad ,$$

with vanishing resistivity. ii) Verify that the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ obeys the equation

$$\nabla^2 \mathbf{B} - e^2 v^2 \mathbf{B} = 0 \quad ,$$

implying that magnetic fields decay within distances of order $(ev)^{-1}$ (Meissner effect).

Problem 8

Consider an $SU(2)$ gauge field coupled to a scalar field ϕ in the spinor representation of the gauge group. Let ϕ acquire a nonzero vacuum expectation value,

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \quad .$$

a) Show that the theory describes three massive gauge bosons, and determine their mass. b) Show that the analogous model with ϕ in the vector representation of $SU(2)$ leads to both massive and massless gauge bosons.

Problem 9

(a) Write down the functional derivative $\delta G/\delta\alpha$ for the R_ξ gauge fixing function G^a in spontaneously broken gauge theory

$$G^a = \partial^\mu A^a_\mu - \xi g (T^a v)_j \varphi_j \quad ,$$

where $\delta\alpha$ is the gauge variation, A^a_μ is the gauge field, v_j are the scalar-field vacuum expectation values and φ_j are the scalar field shifts $\varphi_j = \phi_j - v_j$.

(b) Discuss the corresponding mass terms in the ghost lagrangian. Verify that in the abelian symmetry case ghosts decouple from gauge fields but are still coupled to the scalars.

(c) Decompose the R_ξ gauge-field propagator along transverse and longitudinal projectors, and verify that the longitudinal component propagates with the same mass as the Goldstone field.