University of Oxford Hilary Term 2012

Nonabelian Quantum Field Theory

Problem set A (due February 16)

P1) Using the Fourier mode expansion for the Dirac field,

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3 2k^0} \left(a_s(k) u_s(k) e^{-ikx} + b_s^\dagger(k) v_s(k) e^{ikx} \right) \quad ,$$

obtain the expression of the hamiltonian

$$H = \int d^3x \, \overline{\psi} \left(-i\gamma^j \partial_j + m \right) \psi$$

in terms of $a_s(k)$, $b_s(k)$, and discuss the role of a and b being anticommuting Grassmann variables.

P2) Let χ be a Weyl spinor field. Show that if χ transforms in the $(1/2, 0)$ representation of the Lorentz group, then $\varepsilon\chi^*$, where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, transforms in the $(0, 1/2)$ representation.

P3) Write Dirac spinor ψ in terms of two left-handed Weyl spinors ξ, χ as

$$\psi = \begin{pmatrix} \xi \\ -\varepsilon\chi^* \end{pmatrix} .$$

a) Obtain the expression for Dirac mass lagrangian in terms of the two left-handed Weyl fields. b) For ψ belonging to representation R of an internal symmetry group, and ξ and χ belonging to complex conjugate representations R and $\overline{R}$, show the invariance of the Dirac mass lagrangian under the internal symmetry.

P4) Consider the theory of a single left-handed Weyl spinor field χ. (a) Write the Majorana mass and kinetic energy lagrangian, and corresponding equations of motion. (b) Consider internal symmetry transformations represented by unitary matrices U on χ

$$\chi \to U\chi .$$

Show that the lagrangian mass term in part (a) is invariant under these transformations only if the field transforms in a real representation of the internal symmetry.

P5) Consider the quadratic part of the action for the gauge field A_μ^a in $n \cdot A = 0$ gauge:

$$S = \frac{1}{2} \int d^4x \; A_\mu^a(x) \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu - \frac{1}{\xi} n^\mu n^\nu \right) A_\nu^a(x) \quad .$$

Show that the propagator in this gauge is given by

$$\widetilde{D}^{(n)\ ab}_{F\ \mu\nu}(k) = \frac{-i\delta^{ab}}{k^2 + i\varepsilon}\left[g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n} + (\xi k^2 + n^2)\frac{k_\mu k_\nu}{(k \cdot n)^2}\right] \quad .$$

P6) Consider the BRST transformations

$$\delta A^a_\mu = \lambda D^{ab}_\mu c^b \quad , \quad \delta\psi = ig\lambda c^a t^a \psi \quad , \quad \delta c^a = -\frac{1}{2}\lambda g f^{abc} c^b c^c \quad ,$$

with λ Grassmann parameter. Let Q be the BRST charge operator. Evaluate the action of Q^2 on the gauge, ghost and matter fields and verify that the charge operator is nilpotent, $Q^2 = 0$.

P7) Verify that the generalized Ward identity

$$\int dx \left[\frac{\delta\Gamma}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta u^{\mu a}} + \frac{\delta\Gamma}{\delta c^a}\frac{\delta\Gamma}{\delta v^a} - \frac{1}{g\xi}\frac{\delta\Gamma}{\delta \bar{c}^a}(\partial^\mu A^a_\mu)\right] = 0$$

can be rewritten in terms of the transformed generating functional Γ',

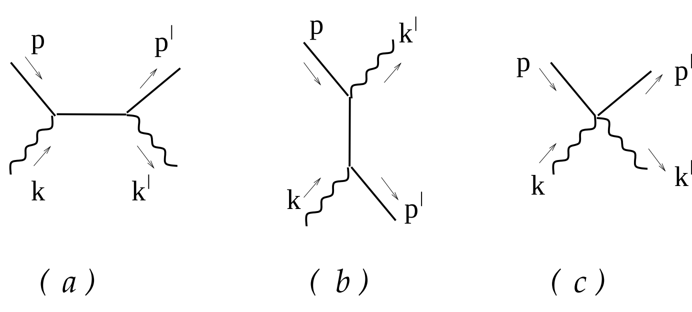
$$\Gamma = \Gamma' - \int dx\ \frac{1}{2\xi}\ (\partial^\mu A^a_\mu)^2 \quad ,$$

in the form

$$\int dx \left[\frac{\delta\Gamma'}{\delta u^{\mu a}}\frac{\delta\Gamma'}{\delta A^a_\mu} + \frac{\delta\Gamma'}{\delta v^a}\frac{\delta\Gamma'}{\delta c^a}\right] = 0 \quad .$$

[Use the equations of motion to relate $\delta\Gamma/\delta\bar{c}$ to the spacetime derivative of $\delta\Gamma/\delta u$.]

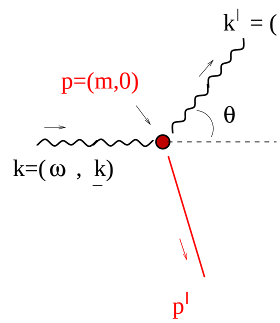
P8) Consider QED Compton scattering of scalar particles:



(i) Determine the scattering matrix element.

(ii) Show that it is gauge invariant.

(iii) Consider the laboratory frame in which the scalar particle is initially at rest:



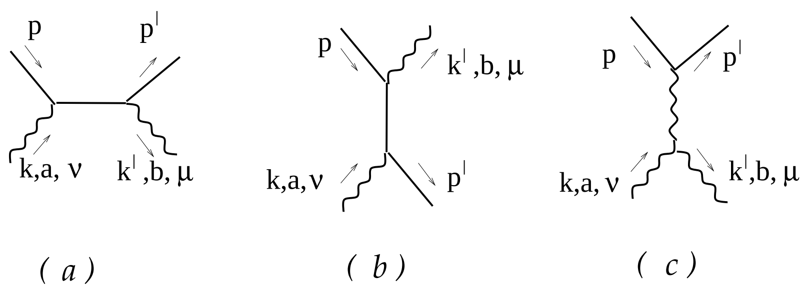
Show that the differential cross section in the solid angle Ω of the final photon momentum, at fixed photon polarizations ε, ε', is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{m^2} \left(\frac{\omega'}{\omega}\right)^2 (\varepsilon \cdot \varepsilon')^2 \quad ,$$

where m is the particle's mass, and ω and ω' are the initial and final photon energies.

(iv) Determine the unpolarized cross section by evaluating the average over polarizations of $(\varepsilon \cdot \varepsilon')^2$.

P9) Consider quark-gluon Compton scattering at lowest order in the QCD coupling g:



(i) Show that the sum of graphs (a) and (b) dotted into k^ν gives

$$M^{(a)+(b)}_{\mu\nu}\ k^\nu = ig^2 [T^a, T^b] \overline{u}(p') \gamma_\mu u(p) \ .$$

(ii) Evaluate the contribution of graph (c) dotted into k^ν,

$$M^{(c)}_{\mu\nu}\ k^\nu \ ,$$

and show that the sum of all graphs gives $M_{\mu\nu} k^\nu = 0$ provided μ is restricted to physical polarizations. Contrast this with the abelian case of electrodynamics.