CP4 REVISION LECTURE ON
<u>WAVES</u>

- ▷ The wave equation.
- ▷ Traveling waves. Standing waves.
- ▷ Dispersion. Phase and group velocities.
- ▷ Boundary effects. Reflection and transmission of waves.

Sept 2004 Q8 Phys & Phil

8. A string of linear density ρ is under tension T, and lies along the x-axis. Derive the wave equation for small transverse displacements $y(x,t)$ of the string. [6]

A finite string of length L lies between $x = a$ and $x = a + L$, and has its ends fixed with $y = 0$. Deduce forms of the initial displacement $y(x, t = 0)$ such that subsequently the displacement $y(x,t)$ retains the same shape, but has a different normalisation $f(t)$ i.e

$$y(x,t) = f(t) \times y(x, t = 0)$$

Find the function $f(t)$ for each of these initial displacements. [7]

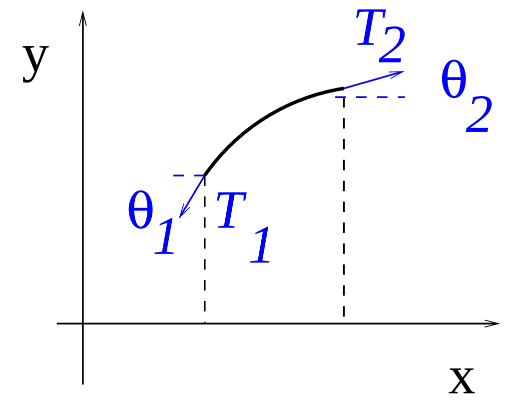
For such a string between $x = a$ and $x = a + L$, the initial displacement is

$$y(x, t = 0) = A \sin(2\pi(x - a)/L) \cos(\pi(x - a)/L)$$

Initially the string is at rest. Determine the subsequent displacement of the string. [7]

(a)

- uniform linear density ρ
- small displacements y



- $T_1 \cos\theta_1 = T_2 \cos\theta_2$

  for small θ, $\cos\theta \simeq 1 \;\Rightarrow\; T_1 = T_2 = T$

- $\rho \; \delta x \; \dfrac{\partial^2 y}{\partial t^2} = T \sin\theta_2 - T \sin\theta_1$

$$\sin\theta \simeq \tan\theta \simeq \frac{\partial y}{\partial x} \quad \Longrightarrow \quad \rho \; \delta x \; \frac{\partial^2 y}{\partial t^2} = T \underbrace{\left[(\frac{\partial y}{\partial x})_2 - (\frac{\partial y}{\partial x})_1 \right]}_{(\partial^2 y/\partial x^2)\delta x + \dots}$$

$$\text{Thus} \qquad \rho \, \frac{\partial^2 y}{\partial t^2} = T \, \frac{\partial^2 y}{\partial x^2}$$

$$\text{i.e.,} \quad \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad , \quad v^2 \equiv \frac{T}{\rho} \qquad \text{wave equation}$$

(b)

- Consider general solution with separated variables:

$$y(x,t) = \left(A\cos k(x-a) + B\sin k(x-a)\right)\left(C\cos kvt + D\sin kvt\right) \quad , \quad v = \sqrt{T/\rho}$$

$$\bullet \text{ Fixed ends : } \quad y(x=a,t) = y(x=a+L,t) = 0 \implies A = 0\,, k = n\pi/L$$

$$\implies y_n(x,t) = \sin[n\pi(x-a)/L]\left(C_n\cos(n\pi vt/L) + D_n\sin(n\pi vt/L)\right) \quad \text{normal modes}$$

$$\text{general solution : } \quad y(x,t) = \sum_n y_n(x,t)$$

- Then, if the initial displacement is of the form $y(x,0) = C\sin[n\pi(x-a)/L]$ for some integer n, subsequent displacements $y(x,t)$ retain the same shape, but with a different time-dependent normalisation $f(t) = C\cos(n\pi vt/L) + D\sin(n\pi vt/L)$

- Normal modes of the string with fixed ends:

$$y_n(x,t) = \sin[n\pi(x-a)/L]\,(C_n \cos(n\pi vt/L) + D_n \sin(n\pi vt/L)) \quad , \quad v = \sqrt{T/\rho}$$

(c)

- String initially at rest $\Rightarrow$ $\partial y/\partial t(x,0) = 0$ $\Rightarrow$ $D_n = 0$

- Initial displacement $y(x,0) = A \sin[2\pi(x-a)/L] \cos[\pi(x-a)/L]$

can be written as $y(x,0) = \frac{1}{2} A \sin[3\pi(x-a)/L] + \frac{1}{2} A \sin[\pi(x-a)/L]$.

Therefore the displacement of the string at subsequent times is given by

$$y(x,t) = \frac{1}{2} A \sin[3\pi(x-a)/L] \cos(3\pi vt/L) + \frac{1}{2} A \sin[\pi(x-a)/L] \cos(\pi vt/L)$$

6. A uniform string is stretched along the x-axis between fixed endpoints at $x = 0$ and $x = D$. If the speed of transverse waves on the string is c, find the wavenumbers and associated frequencies of the standing waves. Write down the precise functional form of $y(x, t)$ for the two lowest frequency modes. [5]

June 2011

- Standing waves with speed c:

$$y(x,t) = (\alpha \cos kx + \beta \sin kx)\,(\gamma \cos ckt + \delta \sin ckt)\ \ ,\ \ \omega = ck$$

- Fixed endpoints:

$$y(x=0,t) = 0 \implies \alpha\ (\gamma \cos ckt + \delta \sin ckt) = 0 \text{ for any } t \implies \alpha = 0$$

$$y(x=D,t) = 0 \implies \beta \sin kD = 0 \implies kD = n\pi\ \ ,\ \ n \text{ integer}$$

$\implies$ wavenumbers and frequencies: $k_n = n\pi/D\,,\ \ \omega_n = n\pi c/D$

- General solution : $y(x,t) = \sum_n y_n(x,t)$ where

$$y_n(x,t) = \sin \frac{n\pi x}{D} \left(\gamma_n \cos \frac{n\pi ct}{D} + \delta_n \sin \frac{n\pi ct}{D} \right)$$

- Two lowest modes:

$$n = 1 \quad : \qquad y_1(x,t) = \sin \frac{\pi x}{D} \left(\gamma_1 \cos \frac{\pi ct}{D} + \delta_1 \sin \frac{\pi ct}{D} \right)$$

$$n = 2 \quad : \qquad y_2(x,t) = \sin \frac{2\pi x}{D} \left(\gamma_2 \cos \frac{2\pi ct}{D} + \delta_2 \sin \frac{2\pi ct}{D} \right)$$

7. The properties of a string are altered so that the wave equation describing small amplitude transverse waves on the string becomes

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2 \frac{\partial^2 y(x,t)}{\partial x^2} = -\mu^2 y(x,t)\,.$$

By utilizing the *ansatz*, $y(x,t) = \mathrm{Re}[\exp i(\omega t \pm kx)]$, or otherwise, find the relation that the modified wave equation implies between the wavenumber k and angular frequency ω for a string of infinite extent. Compute the phase velocity v_p and group velocity v_g of the waves as a function of wavenumber, and comment on the relation of these to c. What are the limiting behaviours of both v_p and v_g as $k \to 0$ and $k \to \infty$? [5]

$\Diamond$ Substituting the ansatz $e^{i(\omega t \pm kx)}$ into the equation gives

$$-\omega^2 + c^2 k^2 = -\mu^2$$

$$\text{i.e.,} \quad \omega^2 = c^2 k^2 + \mu^2$$

• Phase velocity : $\quad v_p = \dfrac{\omega}{k} = \dfrac{\sqrt{c^2 k^2 + \mu^2}}{k} = c\,\sqrt{1 + \mu^2/(c^2 k^2)}$

• Group velocity : $\quad v_g = \dfrac{\partial \omega}{\partial k} = \dfrac{c^2 k}{\sqrt{c^2 k^2 + \mu^2}} = \dfrac{c}{\sqrt{1 + \mu^2/(c^2 k^2)}}$

$$\Longrightarrow \quad v_p v_g = c^2$$

$$\text{with} \quad v_p > c\,, \quad v_g < c$$

$\Diamond$ For $\quad k \rightarrow 0\,, \quad v_p \sim \dfrac{\mu}{k} \rightarrow \infty\,; \quad v_g \sim \dfrac{c^2 k}{\mu} \rightarrow 0$

$\Diamond$ For $\quad k \rightarrow \infty\,, \quad v_p \rightarrow c\,; \quad v_g \rightarrow c$

September 2009

9. Consider the superposition of two travelling waves of equal amplitude with closely spaced angular frequencies and wave numbers, $\Delta\omega = \omega_1 - \omega_2$ and $\Delta k = k_1 - k_2$, respectively. Show that the resultant wave exhibits beats, and sketch the waveform. Explain the significance of the phase and group velocities, v_p and v_g, respectively. [6]

Show that for waves travelling through a dispersive medium,

$$v_g = v_p - \lambda \frac{dv_p}{d\lambda} \,,$$

where λ is the wavelength in the medium. [6]

Waves propagate through a medium and are characterized by the dispersion relation

$$v_p^2 = c^2 + \lambda^2 \omega_0^2 \,,$$

where ω_0 is a constant and c is the speed of light. Show that the product of the phase and group velocities is c^2, and comment on the physical significance of the values of v_p and v_g with respect to c.

A wave travels with $v_g = 0.9\,c$ at $\lambda = 350\,\text{nm}$. Calculate the change in group velocity when λ decreases by $0.02\,\text{nm}$. [8]

(a) • Superposition of travelling waves $y_1(x,t)$ and $y_2(x,t)$:

$$y_1 = A\sin[(k+\Delta k/2)x - (\omega+\Delta\omega/2)t] \ , \quad y_2 = A\sin[(k-\Delta k/2)x - (\omega-\Delta\omega/2)t]$$

• Using $\sin\alpha + \sin\beta = 2\sin[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$, the resultant wave is

$$y = y_1 + y_2 = 2A\sin(kx-\omega t)\cos[(\Delta k \ x - \Delta\omega \ t)/2]$$

♠ 1st factor $\sin$ varies with frequency ω and wave number k,
i.e., close to the original waves y_1 and y_2,
and corresponding speed $v = \omega/k$ (phase velocity).
♠ 2nd factor $\cos$ varies much more slowly,
with frequency $\Delta\omega/2$ and wave number $\Delta k/2$
$\Rightarrow$ amplitude modulation, moving at speed $v_g = \Delta\omega/\Delta k$ (group velocity).
The modulating envelope encloses a group of short waves.

$$\text{For } \ \Delta\omega \, , \ \Delta k \to 0 \, , \quad v_g = \frac{d\omega}{dk}$$

(b)

- Waves travelling through a dispersive medium:

$$v_g = \frac{d\omega}{dk} = \frac{d(vk)}{dk} = v + k\frac{dv}{dk} = v + \frac{2\pi}{\lambda}\frac{dv}{d\lambda}\frac{1}{dk/d\lambda}$$

$$= v + \frac{2\pi}{\lambda}\frac{dv}{d\lambda}\left(-\frac{\lambda^2}{2\pi}\right) = v - \lambda\frac{dv}{d\lambda}$$

(c)

- Suppose the dispersion relation $v = v(\lambda)$ is given by

$$v^2 = c^2 + \lambda^2\omega_0^2$$

$$\text{Then } 2v\,dv = 2\lambda\,d\lambda\,\omega_0^2\ , \quad i.e.\quad \frac{dv}{d\lambda} = \frac{\lambda\omega_0^2}{v}$$

$$\text{So } v_g = v - \lambda\frac{dv}{d\lambda} = v - \frac{\lambda^2\omega_0^2}{v}\ , \quad i.e.\quad v_g = v - \frac{v^2 - c^2}{v} = \frac{c^2}{v} \implies v_g v = c^2$$

$$v_g = \frac{c^2}{v} \;\Rightarrow\; \delta v_g = -\frac{c^2}{v^2}\frac{dv}{d\lambda}\;\delta\lambda = -\frac{c^2}{v^2}\frac{v_g - v}{\lambda}\;\delta\lambda$$

$$v_g = xc \;\Rightarrow\; v = \frac{c}{x}$$

$$\text{So}\;\; \delta v_g = -x^2 c\left(x - \frac{1}{x}\right)\frac{\delta\lambda}{\lambda} = cx(1-x^2)\frac{\delta\lambda}{\lambda}$$

$$x = 0.9\;\;, \qquad \lambda = 350\,nm\;\;, \qquad \delta\lambda = 0.02\,nm \;\;\Longrightarrow\;\; \delta v_g = 9.77\,10^{-6}\,c$$

June 2008

3. Find all possible solutions to the equation

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0$$

of the form $u = g(t)f(x)$, where c is a real constant.

$$u = g(t)f(x) \implies f\,\frac{\partial g}{\partial t} + c\,g\,\frac{\partial f}{\partial x} = 0$$

$$\text{Then} \quad \underbrace{\frac{1}{g}\,\frac{\partial g}{\partial t}}_{function\ of\ t} = \underbrace{-c\,\frac{1}{f}\,\frac{\partial f}{\partial x}}_{function\ of\ x} = K \quad constant$$

$$\implies \quad g = g_0\,e^{Kt} \quad , \qquad f = f_0\,e^{-Kx/c}$$

$$\text{So} \ \ u = \underbrace{g_0 f_0}_{A}\ e^{K(t-x/c)}$$

6. The propagation of transverse waves on a stretched string is described by the wave equation

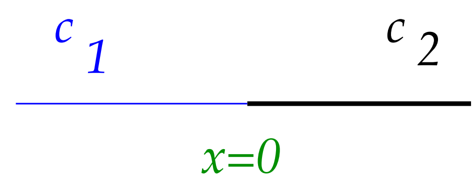
$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

where z is the transverse displacement at point x at time t and c is the speed of propagation.

A string is made of two semi-infinite pieces joined at the origin. For $x < 0$, the speed is c_1; for $x > 0$, the speed is c_2. The wave $z = \cos(\omega t - k_1 x)$ is incident on the boundary, where $k_1 = \omega/c_1$. Find the amplitudes of the reflected and the transmitted waves. [6]

$$k_1 = \omega / c_1$$

$$k_2 = \omega / c_2$$



$$z_I = \text{Re } e^{i(\omega t - k_1 x)} \quad ; \qquad z_T = \text{Re } \left(t e^{i(\omega t - k_2 x)} \right) \quad ; \qquad z_R = \text{Re } \left(r e^{i(\omega t + k_1 x)} \right)$$

- continuity of z at $x = 0$: $z_I + z_R = z_T \implies 1 + r = t$
- continuity of $\partial z / \partial x$ at $x = 0 \implies -ik_1 + ik_1 r = -ik_2 t$

$$\text{Hence} \quad -ik_1(1 - r) = -ik_2(1 + r) \implies r = \frac{k_1 - k_2}{k_1 + k_2} \quad \text{reflected amplitude}$$

$$t = 1 + r = \frac{2k_1}{k_1 + k_2} \quad \text{transmitted amplitude}$$

Note

♠ The continuity of z and $\partial z/\partial x$ at the boundary $x = 0$
in the previous problem
determines the amplitudes of the reflected and transmitted waves
in terms of the amplitude of the incident wave
as a function of the wave numbers k_1 and k_2:

$$r = \frac{k_1 - k_2}{k_1 + k_2} \quad ; \quad t = \frac{2k_1}{k_1 + k_2}$$

♠ The energy transport across the boundary is described by the coefficients
(see next problem)

$$R \equiv \frac{reflected\ \ flux}{incident\ \ flux} = r^2 \qquad \text{reflection coefficient}$$

$$T \equiv \frac{transmitted\ \ flux}{incident\ \ flux} = \frac{k_2}{k_1}\ t^2 \qquad \text{transmission coefficient}$$

The relation $1 = R + T$ expresses energy conservation.

Sept 2003 Q9 Phys

9. A long string lies along the x-axis and is under tension T. The displacement of the string from its equilibrium position at x is given by $y(x,t)$. By considering the forces acting on an element of the string show that y satisfies the wave equation. [5]

Two long strings of different densities ρ_1 and ρ_2 are joined together at $x = 0$. Write down the boundary conditions which must hold at $x = 0$ and use these to show that the power reflection and transmission coefficients R and T, respectively, for a wave incident on the boundary are

$$R = \left(\frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}\right)^2,$$

$$T = \frac{4\sqrt{\rho_1}\sqrt{\rho_2}}{(\sqrt{\rho_1} + \sqrt{\rho_2})^2}.$$ [12]

What is the phase difference between the incident and reflected waves when ρ_1 is less than ρ_2? [3]

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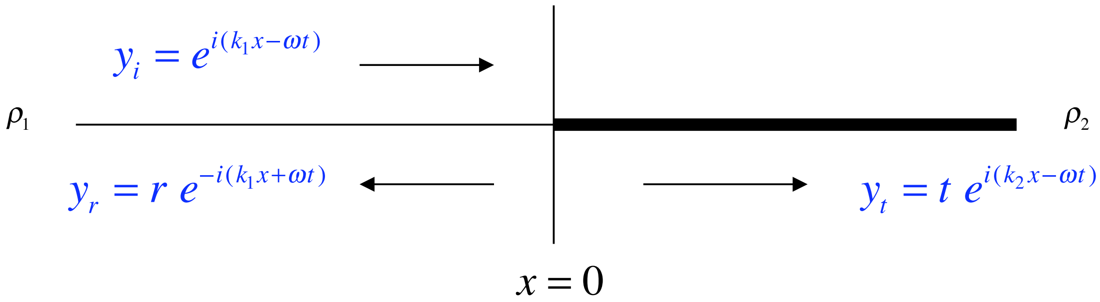

$$v_{1,2} = \sqrt{\frac{T}{\rho_{1,2}}} = \frac{\omega}{k_{1,2}}$$

$$\Rightarrow \quad k_{1,2} = \omega\sqrt{\frac{\rho_{1,2}}{T}}$$

(tension constant in string)

Boundary conditions

$$y_1(x=0,t) = y_2(x=0,t) \qquad \text{(String continuous)}$$

$$\frac{\partial y_1}{\partial t}(x=0,t) = \frac{\partial y_2}{\partial t}(x=0,t) \qquad \text{(Forces continuous if no mass at join)}$$

$$1 + r = t$$

$$k_1(1-r) = k_2\, t$$

$$\Longrightarrow \quad r = \frac{1 - k_2/k_1}{1 + k_2/k_1} = \frac{1 - \sqrt{\rho_2/\rho_1}}{1 + \sqrt{\rho_2/\rho_1}}, \qquad t = \frac{2}{1 + k_2/k_1} = \frac{2}{1 + \sqrt{\rho_2/\rho_1}}$$

Power transmission

$$P_{1,2} = \frac{1}{2} T \omega k_{1,2} |A|^2 \quad \Rightarrow$$

$$R = \frac{k_1}{k_1}|r|^2 = \left(\frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}\right)^2 \qquad T = \frac{k_2}{k_1}|t|^2 = \sqrt{\frac{\rho_2}{\rho_1}} \frac{4}{\left(1 + \sqrt{\rho_2/\rho_1}\right)^2} = \frac{4\sqrt{\rho_1}\sqrt{\rho_2}}{\left(\sqrt{\rho_1} + \sqrt{\rho_2}\right)^2}$$

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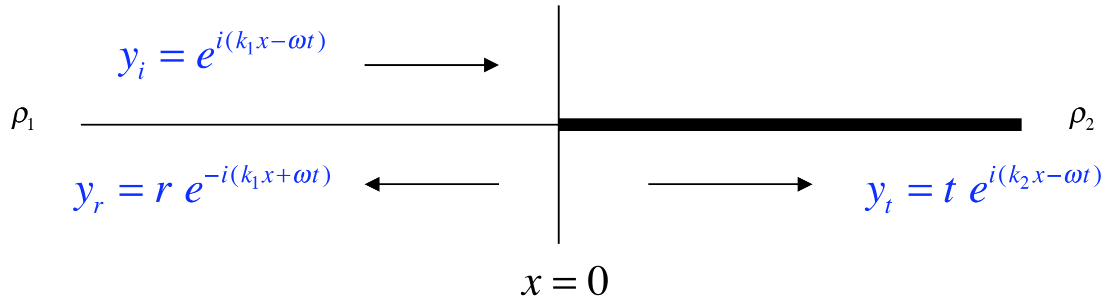


$$\mathrm{v}_{1,2} = \sqrt{\frac{T}{\rho_{1,2}}} = \frac{\omega}{k_{1,2}}$$

$$\Rightarrow \quad k_{1,2} = \omega \sqrt{\frac{\rho_{1,2}}{T}}$$

(tension constant in string)

Boundary conditions

$$y_1(x=0,t) = y_2(x=0,t) \qquad \text{(String continuous)}$$

$$\frac{\partial \mathrm{y}_1}{\partial \mathrm{t}}(x=0,t) = \frac{\partial \mathrm{y}_2}{\partial \mathrm{t}}(x=0,t) \qquad \text{(Forces continuous if no mass at join)}$$

$$1 + r = t$$
$$k_1(1-r) = k_2\, t$$

$$\Longrightarrow \quad r = \frac{1 - k_2/k_1}{1 + k_2/k_1} = \frac{1 - \sqrt{\rho_2/\rho_1}}{1 + \sqrt{\rho_2/\rho_1}}, \qquad t = \frac{2}{1 + k_2/k_1} = \frac{2}{1 + \sqrt{\rho_2/\rho_1}}$$

Phase difference between incident and reflected waves

$$r = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = |r| e^{i\phi},$$

r is negative for $\rho_1 < \rho_2$ *i.e.* $\phi = \pi$

11. A long uniform string is stretched along the x-axis, has a mass density μ, and is held under tension T. A transverse wave of displacement $y(x,t)$ travels along the string.

(a) The wave equation for transverse waves propagating along the string may be

June 2010 written as

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y(x,t)}{\partial t^2} \ .$$

Show that a Gaussian pulse propagating along the string, $A\exp\left[-(x+vt)^2/x_0^2\right]$, where A, v, and x_0 are constants, is a solution of the above differential equation. Sketch the pulse at two different locations corresponding to times t_1 and $t_2 > t_1$. [6]

(b) Show that the wave energy density along the string is given by

$$u = \frac{1}{2}\left\{\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right\} \ .$$

[5]

Hence show that for a sinusoidal wave of amplitude A and angular velocity ω travelling along the string, the energy per wavelength is given by

$$E_\lambda = \frac{1}{2}\mu A^2\omega^2 \ .$$

[5]

(c) Assume now that the string extends to infinity in the $-x$ direction, but the other end is terminated by a mass m at $x = 0$ which is free to move in the y-direction. Calculate the fraction of the energy reflected when a sinusoidal wave propagates in the $+x$ direction. What happens to the rest of the energy? [4]

(a) $$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

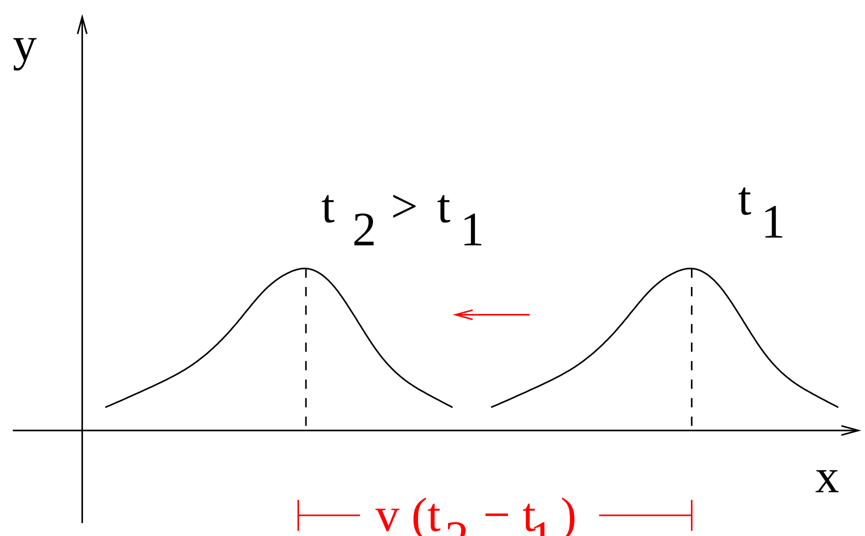
◊ For $y(x,t) = A\exp\left[-(x+vt)^2/x_0^2\right]$ one has

$$\frac{\partial^2 y}{\partial x^2} = -2\frac{y}{x_0^2} + 4(x+vt)^2\frac{y}{x_0^4}$$

and

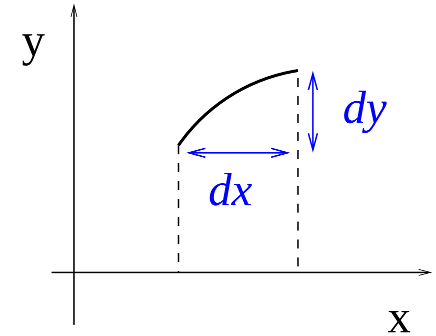
$$\frac{\partial^2 y}{\partial t^2} = -2v^2\frac{y}{x_0^2} + 4(x+vt)^2 v^2\frac{y}{x_0^4} = v^2\frac{\partial^2 y}{\partial x^2} \ .$$

Thus $y(x,t)$ is solution provided $v^2 = T/\mu$.



(b)

- linear mass density μ
- tension T
- transverse displacements y



- Kinetic energy of element dx : $dK = \frac{1}{2}\ \mu\ dx\ \left(\frac{\partial y}{\partial t}\right)^2$

$$\Rightarrow \quad \text{kinetic energy density :}\ \frac{dK}{dx} = \frac{1}{2}\ \mu\ \left(\frac{\partial y}{\partial t}\right)^2$$

- Potential energy of element dx : $dV = T\left(\sqrt{(dx)^2 + (dy)^2} - dx\right)$

$$= T\left(dx\ \underbrace{\sqrt{1 + (\partial y/\partial x)^2}}_{1+(1/2)(\partial y/\partial x)^2+\ldots} - dx\right) = \frac{1}{2}\ T\ dx\ \left(\frac{\partial y}{\partial x}\right)^2$$

$$\Rightarrow \quad \text{potential energy density :}\ \frac{dV}{dx} = \frac{1}{2}\ T\ \left(\frac{\partial y}{\partial x}\right)^2$$

Energy density $\quad u = \frac{dK}{dx} + \frac{dV}{dx} = \frac{1}{2}\ \mu\ \left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}\ T\ \left(\frac{\partial y}{\partial x}\right)^2$

Energy density $\quad u = \frac{dK}{dx} + \frac{dU}{dx} = \frac{1}{2}\,\mu\,\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}\,T\,\left(\frac{\partial y}{\partial x}\right)^2$

◊ For $y(x,t) = A\sin(\omega t - kx)$, $v = \omega/k = \sqrt{T/\mu}$, one has

$$\frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx) \quad , \quad \frac{\partial y}{\partial x} = -kA\cos(\omega t - kx)$$

- kinetic energy per wavelength λ :

$$\frac{1}{\lambda}\int_0^\lambda dx\,\frac{dK}{dx} = \frac{1}{2\lambda}\,\mu\,\omega^2\,A^2 \underbrace{\int_0^\lambda dx\,\cos^2(\omega t - kx)}_{\lambda/2} = \frac{1}{4}\,\mu\,\omega^2\,A^2$$

- potential energy per wavelength λ :

$$\frac{1}{\lambda}\int_0^\lambda dx\,\frac{dV}{dx} = \frac{1}{2\lambda}\,T\,k^2\,A^2 \underbrace{\int_0^\lambda dx\,\cos^2(\omega t - kx)}_{\lambda/2} = \frac{1}{4}\,T\,\frac{\omega^2}{v^2}\,A^2 = \frac{1}{4}\,\mu\,\omega^2\,A^2$$

So $\quad E_\lambda = \frac{1}{\lambda}\int_0^\lambda dx\,\left(\frac{dK}{dx} + \frac{dV}{dx}\right) = \frac{1}{4}\,\mu\,\omega^2\,A^2 + \frac{1}{4}\,\mu\,\omega^2\,A^2 = \frac{1}{2}\,\mu\,\omega^2\,A^2$

(c)

- Terminating mass m at $x = 0$:

$$m\frac{\partial^2 y}{\partial t^2} = -T\ \frac{\partial y}{\partial x}$$

$$y(x,t) = e^{i(\omega t - kx)} + r e^{i(\omega t + kx)}$$

$$\implies \quad -m\omega^2(1+r) = ikT(1-r)$$

$$\text{i.e.} \quad r = \frac{kT - im\omega^2}{kT + im\omega^2}$$

$$R = |r|^2 = 1$$

all energy reflected with phase change 2ϕ, $\tan\phi = m\omega^2/Tk$

phase: $+1$ if $m = 0$, $e^{i\pi} = -1$ if $m \to \infty$