CP4 REVISION LECTURE: NORMAL MODES
OUTLINE

- ▷ Systems of Linear ODEs
- ▷ Solution by Normal Coordinates and Normal Modes
- ▷ Applications to Coupled Oscillators

Introduction to Normal Modes

• Consider a physical system with N degrees of freedom whose dynamics is described by a set of coupled linear ODEs.

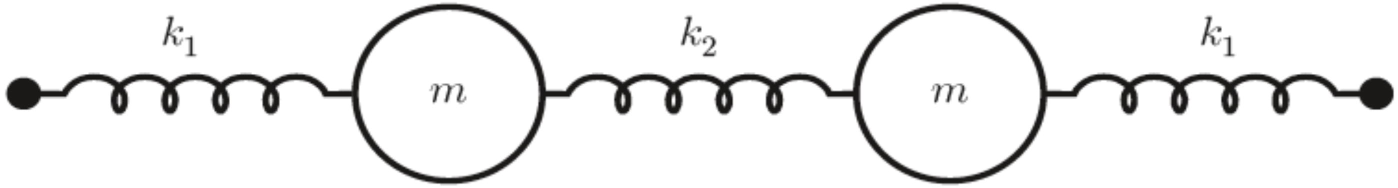
• To determine the *normal modes* of the system means to find a set of N coordinates (*normal coordinates*) describing the system which evolve *independently* like N harmonic oscillators.

• The frequencies of such harmonic motions are the *normal frequencies* of the system.

▷ normal modes describe "collective" motion of the system
▷ general solution expressible as linear superposition of normal modes

June 2003 Q10 Phys

10. A linear mechanical system is constrained to move along a straight line. It consists of two identical masses m and three springs lying on a smooth table, as shown.



The end springs are fastened to fixed supports. The force constant of the two end springs is k_1 with an equilibrium extension s_1, while that of the middle spring connecting the masses is k_2 with an equilibrium extension s_2. Show that, when the displacements of the two masses from equilibrium are x_1 and x_2 respectively, the equations of motion are

$$\ddot{x}_1 + \frac{(k_1 + k_2)}{m} x_1 - \frac{k_2}{m} x_2 = 0,$$

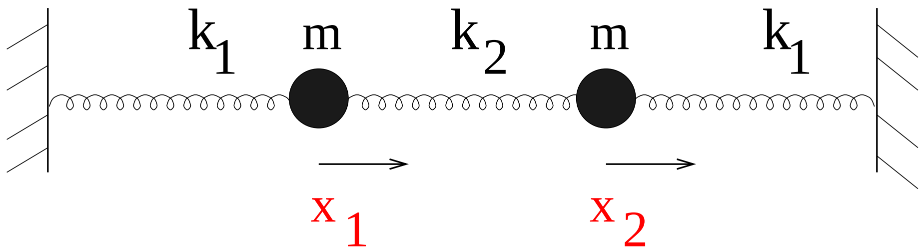
$$\ddot{x}_2 + \frac{(k_1 + k_2)}{m} x_2 - \frac{k_2}{m} x_1 = 0.$$

[6]

Hence determine the frequencies of the two normal modes of the system and show for the case when $k_1 = k_2 = k$ that the two frequencies are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. [10]

What do the two cases $k_1 \gg k_2$ and $k_2 \gg k_1$ represent physically? [4]

$\Diamond$ Two masses m moving on a straight line without friction under the action of three springs:



$$m\ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2)$$

$$m\ddot{x}_2 = -k_1 x_2 + k_2(x_1 - x_2)$$

i.e.,

$$\ddot{x}_1 + [(k_1 + k_2)/m]x_1 - (k_2/m)x_2 = 0$$

$$\ddot{x}_2 + [(k_1 + k_2)/m]x_2 - (k_2/m)x_1 = 0$$

2 coupled 2nd-order linear ODEs

♠ Two calculational approaches to finding normal modes of the system:

- decoupling method
- matrix method

Decoupling method

$$m\ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2)$$

$$m\ddot{x}_2 = -k_1 x_2 + k_2(x_1 - x_2)$$

- Setting

$$q_1 = \frac{x_1 + x_2}{\sqrt{2}} \quad , \quad q_2 = \frac{x_1 - x_2}{\sqrt{2}}$$

normal

coordinates

gives

$$\ddot{q}_1 + (k_1/m) q_1 = 0$$

$$\ddot{q}_2 + [(k_1 + 2k_2)/m] q_2 = 0$$

2 *decoupled* simple harmonic oscillators with frequencies

$$\omega_1 = \sqrt{\frac{k_1}{m}} \quad , \quad \omega_2 = \sqrt{\frac{k_1 + 2k_2}{m}}$$

normal frequencies

- For $k_1 = k_2 = k$, we have $\omega_1 = \sqrt{k/m}$, $\omega_2 = \sqrt{3k/m}$.

Matrix method

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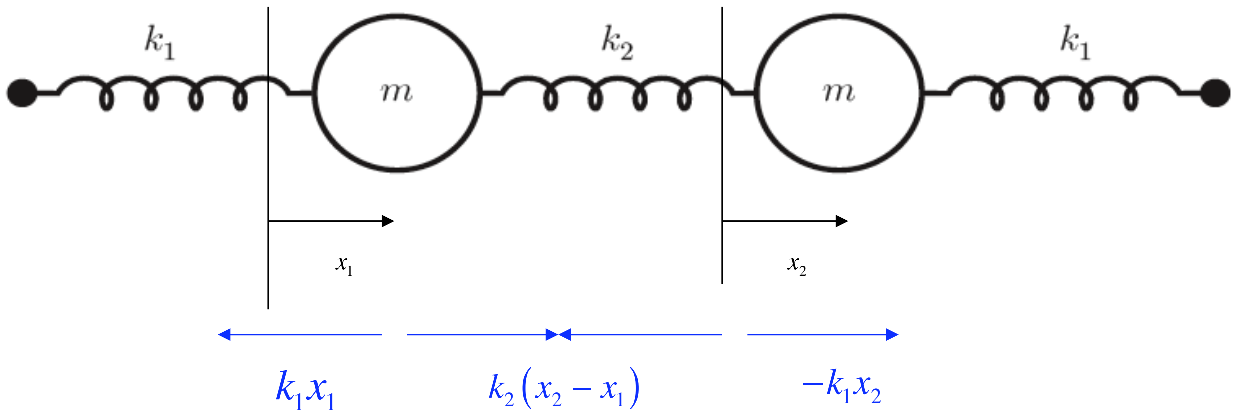

Newton's law for each mass gives:

$$\ddot{x}_1 + \frac{(k_1 + k_2)}{m} x_1 - \frac{k_2}{m} x_2 = 0,$$

$$\ddot{x}_2 + \frac{(k_1 + k_2)}{m} x_2 - \frac{k_2}{m} x_1 = 0.$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{(k_1 + k_2)}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{d^2}{dt^2} + \frac{(k_1 + k_2)}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$

$$\begin{pmatrix} -\omega^2 + \frac{(k_1 + k_2)}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & -\omega^2 + \frac{(k_1 + k_2)}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\text{Try } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t} \qquad \begin{pmatrix} -\omega^2 + \frac{(k_1+k_2)}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & -\omega^2 + \frac{(k_1+k_2)}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvalues

$$Det \begin{pmatrix} -\omega^2 + \frac{(k_1+k_2)}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & -\omega^2 + \frac{(k_1+k_2)}{m} \end{pmatrix} = 0 \quad \Rightarrow \quad \omega_{1,2}^2 = \frac{(k_1+k_2)}{m} \pm \frac{k_2}{m}$$

$$\boxed{k_1 = k_2 = k \quad \Rightarrow \quad \omega_1 = \sqrt{k/m}, \quad \omega_2 = \sqrt{3k/m}}$$

$k_1 \gg k_2$ 2 nearly decoupled oscillators with frequency $\sqrt{\mathrm{k}_1 / m}$

$k_2 \gg k_1$ 1 nearly decoupled oscillator with frequency $\sqrt{2\mathrm{k}_2 / m} + CM$ motion

Remarks

- 2 linear ODEs in $x_1(t), x_2(t) \longrightarrow$ 2 *normal frequencies* ω_1, ω_2 at which the system can oscillate as a whole.

$\Rightarrow q_1 \propto x_1 + x_2$ and $q_1 \propto x_1 - x_2$ oscillate *independently* as 2 SHO's with frequencies ω_1 and ω_2 (*normal modes*)

- any motion of the system will be linear superposition of normal modes:

GS = c_1 NM1 + c_2 NM2

$$x_1(t) = \frac{q_1 + q_2}{\sqrt{2}} = \frac{1}{\sqrt{2}} \, [A \sin(\omega_1 t + \phi) + B \sin(\omega_2 t + \psi)]$$

$$x_2(t) = \frac{q_1 - q_2}{\sqrt{2}} = \frac{1}{\sqrt{2}} \, [A \sin(\omega_1 t + \phi) - B \sin(\omega_2 t + \psi)]$$

- total energy = sum of energies of each normal mode

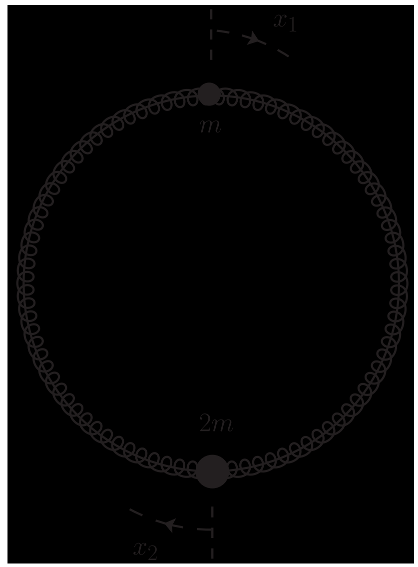
$$E = T + V = \frac{1}{2} \, m \, (\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} \, k_1 x_1^2 + \frac{1}{2} \, k_2 \, (x_2 - x_1)^2 + \frac{1}{2} \, k_1 x_2^2$$

$$= \underbrace{\frac{1}{2} \, m\dot{q}_1^2 + \frac{1}{2} \, m\omega_1^2 q_1^2}_{E_1} + \underbrace{\frac{1}{2} \, m\dot{q}_2^2 + \frac{1}{2} \, m\omega_2^2 q_2^2}_{E_2} = E_1 + E_2$$

Section B

8. Two particles of mass m and $2m$ are free to slide on a frictionless horizontal circular wire of radius r. The particles are connected by two identical massless springs of spring constant k and natural length πr, which also wind around the wire, as shown in the figure.

June 2011



(a) Write down the equations of motion for the displacements x_1 and x_2 of the two particles. [2]

(b) Find the frequencies of the normal modes of the system, and qualitatively explain their values. [6]

(c) Find the functional forms of the motions associated with each of the normal modes, including the appropriate time dependence. [7]

(d) If, at time $t = 0$, the displacements and velocities of the two particles are given by $x_1(0) = \pi r/10$, $x_2(0) = -\pi r/20$, and $\dot{x}_1(0) = \dot{x}_2(0) = \pi \omega r/20$ (where ω is the largest of the frequencies found in (b)), find the first time when the velocity of the particle of mass $2m$ is zero. Write your answer in terms of ω. [5]

(a) Equations of motion:

$$m\ddot{x}_1 = -k(x_1 - x_2) + k(x_2 - x_1)$$

$$2m\ddot{x}_2 = -k(x_2 - x_1) + k(x_1 - x_2)$$

i.e.

$$\ddot{x}_1 = -\frac{2k}{m}(x_1 - x_2)$$

$$\ddot{x}_2 = -\frac{k}{m}(x_2 - x_1)$$

Decoupling method

(b) $\Rightarrow$ $$\ddot{x}_1 + 2\ddot{x}_2 = 0$$

$$\ddot{x}_1 - \ddot{x}_2 = -\frac{3k}{m}(x_1 - x_2)$$

- Set

$$q_1 = x_1 + 2x_2 \quad , \quad q_2 = x_1 - x_2 \qquad \text{normal coordinates}$$

$\Rightarrow$

$$\ddot{q}_1 = 0$$

$$\ddot{q}_2 + (3k/m)q_2 = 0$$

$$\Rightarrow \qquad \omega_1 = 0 \ , \quad \omega_2 = \sqrt{\frac{3k}{m}} \equiv \omega \qquad \text{normal frequencies}$$

$q_1 \propto$ center-of-mass coordinate, $q_2 \propto$ relative coordinate

- $\omega_1 = 0 \Rightarrow$ uniform motion of q_1 round the wire
- q_2 oscillates with frequency $\omega_2 = \sqrt{3k/m} \equiv \omega$

(c) $\Rightarrow$

$$q_1(t) = c_1 t + c_1'$$

$$q_2(t) = c_2 \sin \omega t + c_2' \cos \omega t$$

$\Rightarrow$

$$x_1(t) = \frac{q_1 + 2q_2}{3} = C_1 t + C_1' + 2(C_2 \sin \omega t + C_2' \cos \omega t)$$

$$x_2(t) = \frac{q_1 - q_2}{3} = C_1 t + C_1' - (C_2 \sin \omega t + C_2' \cos \omega t)$$

- linear superposition of normal modes

(d) Initial conditions

$$x_1(0) = \pi r/10\,,\ x_2(0) = -\pi r/20\,,\ \dot{x}_1(0) = \dot{x}_2(0) = \pi\omega r/20 \quad \Rightarrow$$

$$\pi r/10 = C_1' + 2C_2'$$

$$-\pi r/20 = C_1' - C_2'$$

$$\pi\omega r/20 = C_1 + 2\omega C_2$$

$$\pi\omega r/20 = C_1 - \omega C_2$$

$$\Rightarrow\ C_1' = 0\,,\ C_2' = \pi r/20\,,\ C_1 = \pi\omega r/20\,,\ C_2 = 0$$

$$\text{Thus}\quad x_1(t) = \pi r\omega t/20 + (\pi r/10)\cos\omega t$$

$$x_2(t) = \pi r\omega t/20 - (\pi r/20)\cos\omega t$$

$$\dot{x}_2(t_0) = 0 \quad \Rightarrow \quad \pi r\omega/20 + (\pi r/20)\omega\sin\omega t_0 = 0$$

$$\Rightarrow \quad t_0 = 3\pi/(2\omega) = (\pi/2)\sqrt{3m/k}$$

Alternative method to find the normal modes:

Matrix method

$$\ddot{x}_1 = -\frac{2k}{m}(x_1 - x_2)$$

$$\ddot{x}_2 = -\frac{k}{m}(x_2 - x_1)$$

$$\text{Ansatz} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t} \longrightarrow \begin{pmatrix} -\omega^2 + 2k/m & -2k/m \\ -k/m & -\omega^2 + k/m \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 0$$

$$\begin{vmatrix} -\omega^2 + 2k/m & -2k/m \\ -k/m & -\omega^2 + k/m \end{vmatrix} = (-\omega^2 + 2k/m)(-\omega^2 + k/m) - 2(k/m)^2 = 0$$

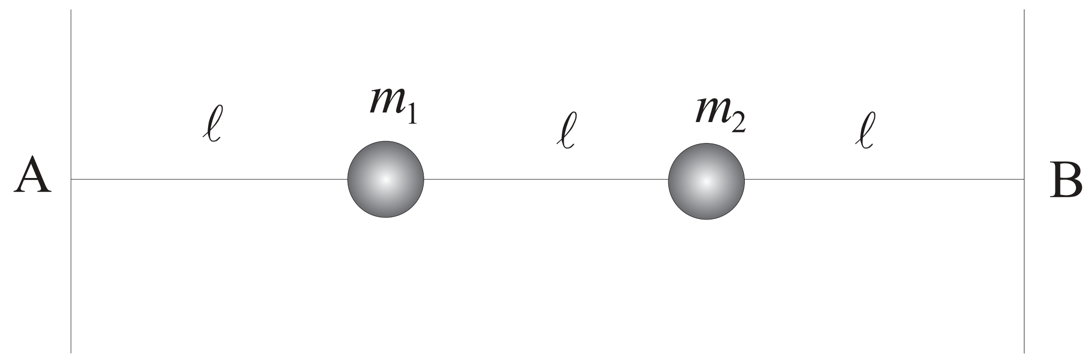
$$\Rightarrow \omega^2(\omega^2 - 3k/m) = 0$$

i.e.,

$$\omega_1^2 = 0 \ , \quad \omega_2^2 = \frac{3k}{m} \qquad \text{normal frequencies}$$

June 2007

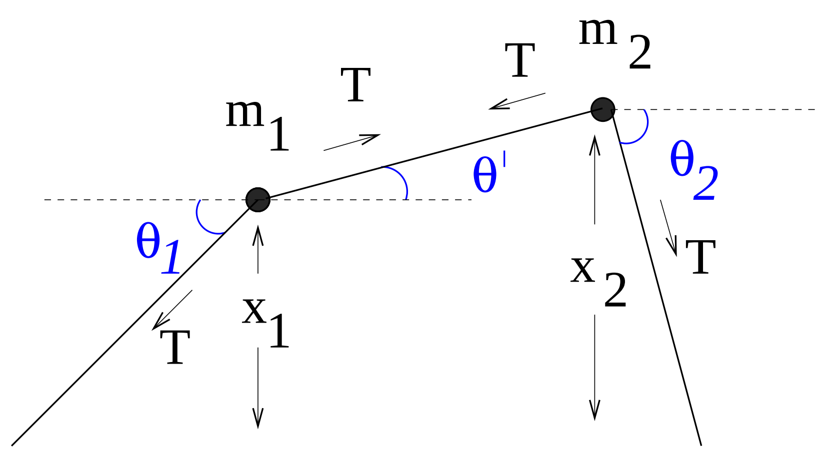
10.



Two masses m_1 and m_2 are connected together by a massless elastic string at tension T between two fixed points A and B, as shown. Write down the equations of motion for x_1 and x_2, the transverse position in one plane from equilibrium of the masses, neglecting the effects of gravity and damping. [4]

Deduce values for the frequencies and related amplitude ratios of the two normal modes for $m_1 = 1.5\,\mathrm{kg}$, $m_2 = 0.8\,\mathrm{kg}$, $\ell = 1\,\mathrm{m}$, $T = 3\,\mathrm{N}$. [8]

The masses are initially in equilibrium at rest. At time $t = 0$ the mass m_2 is given a sudden transverse velocity $0.5\,\mathrm{m\,s^{-1}}$. Find the subsequent displacement of x_1 as a function of time. [8]



◊ Equations of motion for transverse displacements x_1 and x_2:

$$m_1 \ddot{x}_1 = -T x_1/\ell + T(x_2 - x_1)/\ell$$

$$m_2 \ddot{x}_2 = -T x_2/\ell - T(x_2 - x_1)/\ell$$

$$\text{Ansatz} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t} \longrightarrow \begin{pmatrix} -\omega^2 + 2T/(m_1\ell) & -T/(m_1\ell) \\ -T/(m_2\ell) & -\omega^2 + 2T/(m_2\ell) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 0$$

$$\det = 0 \quad \Rightarrow \quad [-\omega^2 + 2T/(m_1\ell)][-\omega^2 + 2T/(m_2\ell)] - (T/\ell)^2/(m_1 m_2) = 0$$

$$\Rightarrow \omega^4 - \frac{2T}{\ell}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\omega^2 + \frac{3}{m_1 m_2}\left(\frac{T}{\ell}\right)^2 = 0$$

i.e.,

$$\omega_{1,2}^2 = \frac{T}{\ell}\frac{1}{m_1 m_2}\left[m_1 + m_2 \pm \sqrt{m_1^2 + m_2^2 - m_1 m_2}\right] \qquad \text{normal frequencies}$$

- $m_1 = 1.5$ Kg, $m_2 = 0.8$ Kg, $\ell = 1$ m, $T = 3$ N
  $\Rightarrow\ \omega_1 = 3\ s^{-1}$, $\omega_2 = 1.58\ s^{-1}$

◊ Amplitude ratios X_2/X_1 in the two normal modes:

$$(-\omega^2 + 2T/(m_1\ell))X_1 - T/(m_1\ell)X_2 = 0$$

$$\Rightarrow\ (X_2/X_1)_{NM1,2} = 2 - m_1\ell\omega_{1,2}^2/T$$

- Putting in the above values of ω_1 and ω_2

$$\Rightarrow \left(\frac{X_2}{X_1}\right)_{NM1} = -\frac{5}{2}\,, \quad \left(\frac{X_2}{X_1}\right)_{NM2} = \frac{3}{4}$$

- The general solution is linear superposition of the two normal modes:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A \begin{pmatrix} 1 \\ -5/2 \end{pmatrix} \cos(\omega_1 t + \phi) + B \begin{pmatrix} 1 \\ 3/4 \end{pmatrix} \cos(\omega_2 t + \psi)$$

◊ Initial conditions $x_1(0) = x_2(0) = 0$, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0.5$ m/s require

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ -5/2 \end{pmatrix} \cos\phi + B \begin{pmatrix} 1 \\ 3/4 \end{pmatrix} \cos\psi$$

$$\begin{pmatrix} 0 \\ 0.5 \end{pmatrix} = -\omega_1 A \begin{pmatrix} 1 \\ -5/2 \end{pmatrix} \sin\phi - \omega_2 B \begin{pmatrix} 1 \\ 3/4 \end{pmatrix} \sin\psi$$

$$\Rightarrow \quad \phi = \psi = -\pi/2\,, \quad A = -2/(13\omega_1)\,, \quad B = 2/(13\omega_2)$$

- Putting in the values of ω_1 and ω_2

$$\Rightarrow \; x_1(t) = -0.051 \;\sin 3t + 0.097 \sin 1.58t$$

Sept 2003 Q10 Phys

10. A double pendulum consists of light, inextensible strings, AB and BC, each of length ℓ. It is fixed at one end A and carries two particles, each of mass m, which hang under gravity. One of the particles is attached at the mid-point B while the other is located at C. The pendulum is constrained to move in a vertical plane. The angle between the vertical and AB is θ while the angle between BC and the vertical is ϕ. Show that, for small angles about the equilibrium position,

$$\ddot{\theta} + \frac{g}{\ell}(2\theta - \phi) = 0,$$

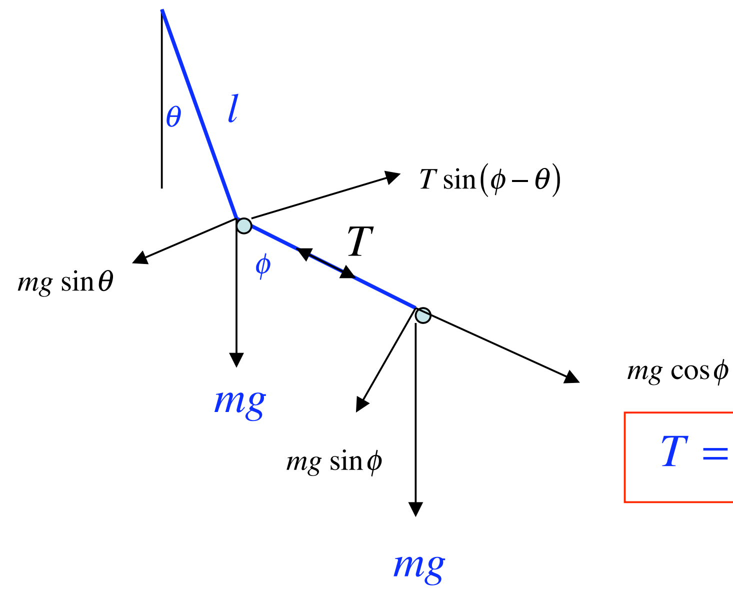
$$\ddot{\phi} + 2\frac{g}{\ell}(\phi - \theta) = 0.$$ [7]

Determine the normal frequencies for small oscillations of this system and show that the higher frequency is $\left(\sqrt{2}+1\right)$ times the lower frequency. Show also that, in both normal modes, the amplitude of ϕ is $\sqrt{2}$ times that of θ. [8]

Draw a sketch to show the instantaneous positions of the two masses at maximum amplitude for both the high and low frequency modes and calculate the difference in the frequency of the two modes for $\ell = 10\,\text{cm}$. [5]

[Take the acceleration due to gravity to be $g = 9{\cdot}8\,\text{ms}^{-2}$.]

Sept 2003 Q10 Phys



$$\boxed{T = mg\cos\phi \simeq mg}$$

$$ml\ddot{\theta} = -mg\sin\theta + mg\sin(\phi-\theta) \simeq -mg(2\theta-\phi)$$

$$\ddot{\theta} + \frac{g}{l}(2\theta-\phi) = 0$$

$$m\left(l\ddot{\theta} + l\ddot{\phi}\right) = -mg\sin\phi \quad \Rightarrow \quad \ddot{\phi} = -\frac{g}{l}\phi + \frac{g}{l}(2\theta-\phi)$$

$$\ddot{\phi} + \frac{2g}{l}(\phi-\theta) = 0$$

$$\boxed{\begin{pmatrix} \frac{d^2}{dt^2} + \frac{2g}{l} & -\frac{g}{l} \\ -\frac{2g}{l} & \frac{d^2}{dt^2} + \frac{2g}{l} \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

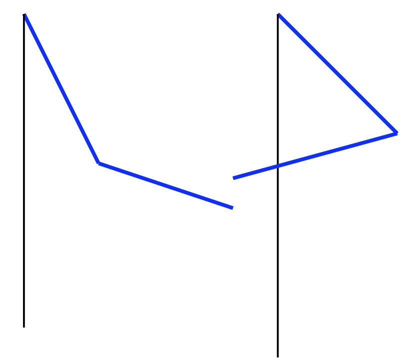
Sept 2003 Q10 Phys

$$\boxed{\begin{pmatrix} \dfrac{d^2}{dt^2}+\dfrac{2g}{l} & -\dfrac{g}{l} \\ -\dfrac{2g}{l} & \dfrac{d^2}{dt^2}+\dfrac{2g}{l} \end{pmatrix}\begin{pmatrix}\theta\\ \phi\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}}$$

$$\text{Try } \begin{pmatrix}\theta\\ \phi\end{pmatrix}=\begin{pmatrix}X\\ Y\end{pmatrix}e^{i\omega t} \qquad \begin{pmatrix} -\omega^2+\dfrac{2g}{l} & -\dfrac{g}{l} \\ -\dfrac{2g}{l} & -\omega^2+\dfrac{2g}{l} \end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Eigenvalues

$$\boxed{Det\begin{pmatrix} -\omega^2+\dfrac{2g}{l} & -\dfrac{g}{l} \\ -\dfrac{2g}{l} & -\omega^2+\dfrac{2g}{l} \end{pmatrix}=0 \quad \Rightarrow \quad \omega_{1,2}^2=\frac{g}{l}\left(2\pm\sqrt{2}\right)}$$



$$\frac{\omega_1^2}{\omega_2^2}=\frac{2+\sqrt{2}}{2-\sqrt{2}}=\frac{2+\sqrt{2}}{2-\sqrt{2}}.\frac{2+\sqrt{2}}{2+\sqrt{2}}=\frac{\left(2+\sqrt{2}\right)^2}{2} \quad \Rightarrow \quad \frac{\omega_1}{\omega_2}=1+\sqrt{2}$$

Eigenvectors

$$\begin{pmatrix} -\dfrac{g}{l}\left(2\pm\sqrt{2}\right)+\dfrac{2g}{l} & -\dfrac{g}{l} \\ -\dfrac{2g}{l} & -\dfrac{g}{l}\left(2\pm\sqrt{2}\right)+\dfrac{2g}{l} \end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \mp\sqrt{2} & -1 \\ -2 & \mp\sqrt{2} \end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\boxed{\frac{Y}{X}=\mp\sqrt{2}}$$

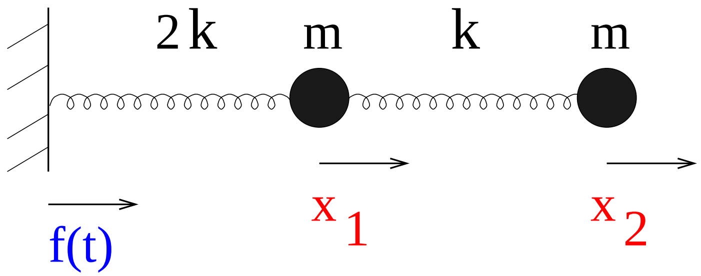
June 2006

12. Two identical masses $m_1 = m_2 = m$ are connected by a massless spring with spring constant k. Mass m_1 is attached to a support by another massless spring with spring constant $2k$. The masses and springs lie along the horizontal x-axis on a smooth surface. The masses and the support are allowed to move along the x-axis only. The displacement of the support in the x-direction at time t is given by $f(t)$ and is externally controlled. Write down a system of differential equations describing the evolution of the displacements x_1 and x_2 of the masses from their equilibrium positions. [5]

Determine the frequencies of the normal modes and their amplitude ratios. [8]

The displacement of the support is given by $f(t) = A\sin(\omega t)$ with $\omega^2 = k/m$ and constant amplitude A. Find expressions for $x_1(t)$ and $x_2(t)$ assuming that any transients have been damped out by a small, otherwise negligible, damping term. [7]

$\Diamond$ Coupled oscillators with a driving term:



$$m\ddot{x}_2 = -k(x_2 - x_1)$$

$$m\ddot{x}_1 = -2k(x_1 - f(t)) - k(x_1 - x_2)$$

- Homogeneous case ($f = 0$):

$$\longrightarrow \quad \text{normal frequencies} \ \ \omega_1 = \sqrt{\frac{k}{m}(2+\sqrt{2})} \ , \quad \omega_2 = \sqrt{\frac{k}{m}(2-\sqrt{2})}$$

$$\text{amplitude ratios}: \quad (X_2/X_1)_{NM1} = 1-\sqrt{2} \ , \quad (X_2/X_1)_{NM2} = 1+\sqrt{2}$$

- Driving term $f(t) = A\sin\omega t$, $\omega = \sqrt{k/m}$:

$$x_1(t) = C_1 \mathrm{Im}\ e^{i\omega t} \quad , \qquad x_2(t) = C_2 \mathrm{Im}\ e^{i\omega t}$$

$$\Longrightarrow \qquad -\omega^2 \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \omega^2 \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \omega^2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} A$$

$$\text{Thus}: \quad -C_1 = -3C_1 + C_2 + 2A$$

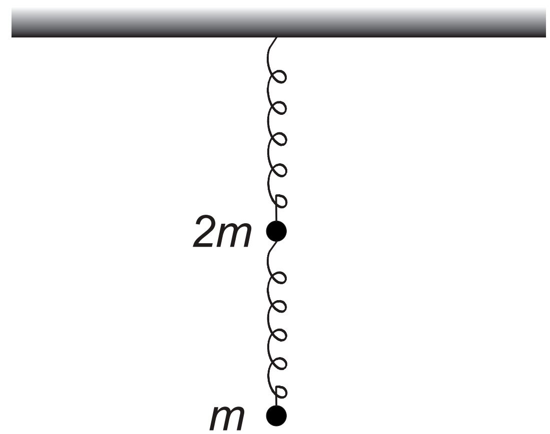
$$-C_2 = C_1 - C_2$$

$$\Longrightarrow \qquad C_1 = 0, C_2 = -2A$$

$$x_1(t) = 0 \quad , \qquad x_2(t) = -2A\sin\omega t$$

8. Two massless springs each have spring constant k. Masses $2m$ and m are attached as shown in the figure.

June 2008



The masses make small vertical oscillations about their equilibirium positions. Show that the respective displacements x and y of the masses $2m$ and m satisfy the coupled differential equations

$$
\begin{aligned}
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{k}{2m}(y-2x) \\
\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{k}{m}(x-y)
\end{aligned}
$$

and explain why there is no term involving the acceleration due to gravity. [7]

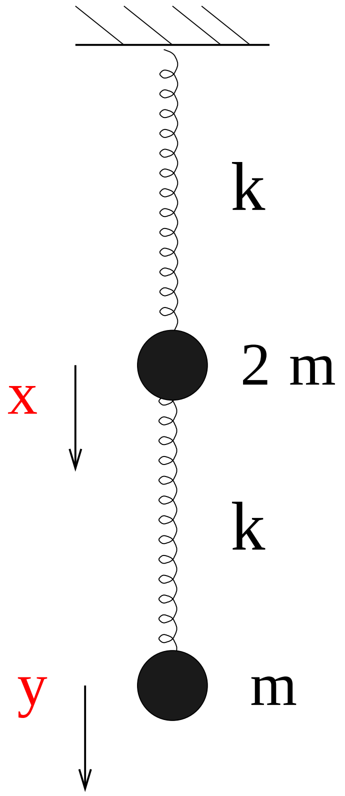
Find expressions for the normal frequencies for small oscillations of the masses. [7]

Find the ratio of the amplitudes for each normal mode. [6]

$$m\ddot{y} = -k(y - x)$$

$$2m\ddot{x} = -kx - k(x - y)$$

g does not appear because x and y are displacements from equilibrium (gravity will determine shift mg/k of the zero)



$$\ddot{y} = (k/m)(x - y)$$

$$\ddot{x} = [k/(2m)](y - 2x)$$

Matrix method

$$\text{Ansatz} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t} \longrightarrow \begin{pmatrix} -\omega^2 + k/m & -k/2m \\ -k/m & -\omega^2 + k/m \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\begin{vmatrix} -\omega^2 + k/m & -k/2m \\ -k/m & -\omega^2 + k/m \end{vmatrix} = (-\omega^2 + k/m)^2 - (k/m)^2/2 = 0$$

$$\Rightarrow \omega^2 - \frac{k}{m} = \pm \frac{1}{\sqrt{2}} \frac{k}{m}$$

i.e.,

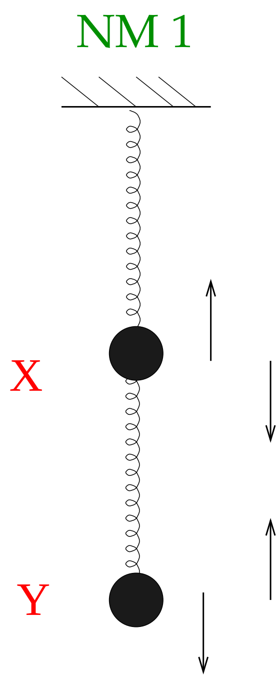
$$\omega^2 = \frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right) \qquad \text{normal frequencies}$$

- Normal mode 1: $\omega^2 = \omega_1^2 = (k/m)\left(1 + 1/\sqrt{2}\right)$

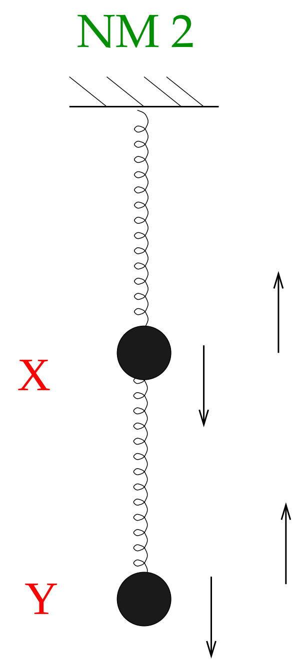
$$(-\omega_1^2 + k/m)X = [k/(2m)]Y \implies -X/\sqrt{2} = Y/2 \quad i.e., \quad X/Y = -1/\sqrt{2}$$

- Normal mode 2: $\omega^2 = \omega_2^2 = (k/m)\left(1 - 1/\sqrt{2}\right)$

$$(-\omega_2^2 + k/m)X = [k/(2m)]Y \implies X/\sqrt{2} = Y/2 \quad i.e., \quad X/Y = 1/\sqrt{2}$$



move against



move together