CP3 REVISION LECTURE ON

Ordinary differential equations

1 First order linear equations

2 First order nonlinear equations

3 Second order linear equations with constant coefficients

4 Systems of linear ordinary differential equations

BASIC CONCEPTS

- Every differential equation involves a differential operator.

Order of a differential operator: order of highest derivative contained in it

- Linearity: differential operator L is *linear* if for any two functions f and g

$$L(\alpha f + \beta g) = \alpha L f + \beta L g$$

with α and β any constants.

- Linearity $\Rightarrow$ superposition principle:

If f_1 and f_2 are solutions of $Lf = 0$, then any linear combination $\alpha f_1 + \beta f_2$ is also solution.

$Lf = 0$ homogeneous differential equation

$Lf = h(x) \neq 0$ inhomogeneous differential equation

• General solution of linear inhomogeneous ODE $Lf = h$ is sum of a particular solution f_0 (the "particular integral", PI) and the general solution f_1 of the associated homogeneous equation (the "complementary function", CF):

$$f = f_0 + f_1 ,$$

$$\text{i.e.,} \quad \text{GS} \;=\; \text{PI} \;+\; \text{CF} .$$

• Initial conditions:

n initial conditions needed to specify solution of linear ODE of order n

First order linear equations

General form : $\frac{\mathrm{d}f}{\mathrm{d}x} + q(x) f = h(x).$

Easy to solve

Integrating factor

Look for a function I(x) such that $\mathrm{I(x)}\frac{\mathrm{d}f}{\mathrm{d}x} + I(x)q(x) f \equiv \frac{dIf}{dx} = I(x)h(x)$

$$I(x) = e^{\int^{x} q(x')\,dx'}$$

Solution : $f(x) = \frac{1}{I(x)}\int_{x_0}^{x} I(x')h(x')dx'$

June 2006

2. Find the general solution of the differential equation

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \sin x$$

[Answ.: $x(c - \cos x)$]

June 2008

4. Solve the differential equation

$$x(x+1)\frac{dy}{dx} + y = x(x+1)^2 e^{-x^2}$$

[Answ.: $(c - e^{-x^2}/2)(x+1)/x$]

September 2009

3. Solve the differential equation

$$x\frac{dy}{dx} + 2y = \cos x$$

[Answ.: $(\cos x + x\sin x + c)/x^2$]

First order nonlinear equations

Although no general method for solution is available, there are several cases of physically relevant nonlinear equations which can be solved analytically :

Separable equations

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Solution : $\int g(y)dy = \int f(x)dx$

Almost separable equations

$$\frac{dy}{dx} = f(ax + by)$$

Change variables : $z = ax + by$ $\qquad \frac{dz}{dx} = a + bf(z)$ Separable

Homogeneous equations

$$\frac{dy}{dx} = f(y/x).$$

Change variables : $y = vx$ $\qquad \frac{dv}{dx} = \frac{1}{x}(f(v) - v)$ Separable

Homogeneous but for constants

$$\frac{dy}{dx} = \frac{x+2y+1}{x+y+2}$$

Change variables : $x = x' + a,\ y = y' + b$

$$\frac{dy'}{dx'} = \frac{x'+2y'+1+a+2b}{x'+y'+2+a+b} = \frac{x'+2y'}{x'+y'},$$ $a=-3,\ b=1$ Homogeneous

The Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \qquad n \neq 1$$

Change variables : $z = y^{1-n}$

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x),$$ First order linear

Exact equations : $$\frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$$

for a given function $\phi(x, y)$

- Then solution is determined by

$$\phi(x, y) = \text{constant} \ .$$

June 2005

1. Solve the differential equation

$$2\,\frac{dy}{dx} = \frac{y(x+y)}{x^2} \qquad \text{“homogeneous”}$$

[Answ.: $y = x/(1 - c\sqrt{x})$]

June 2007

8. Find the general solution to the differential equation

$$y\,\frac{dy}{dx} = \frac{x}{4x+3} \qquad \text{“separable”}$$

[Answ.: $y^2 = c + x/2 - (3/8)\ln(4x+3)$]

September 2009

3. Solve the differential equation

$$\frac{dy}{dx} = \frac{x+y}{1-x-y} \qquad \text{“almost separable”}$$

[Answ.: $y - (x+y)^2/2 = c$]

1. Solve the differential equation

$$\frac{dy}{dx} + xy = xy^2 \quad \text{“Bernoulli”}$$

[Answ.: $y = 1/(1 + ce^{x^2/2})$]

2. Solve the differential equation

$$(2y - x\cos y)\,\frac{dy}{dx} = x + \sin y \quad \text{“exact”}$$

[Answ.: $x\sin y - y^2 + x^2/2 = c$]

GEOMETRICAL INTERPRETATION OF SOLUTIONS

• General solution of a first-order ODE $y' = f(x, y)$

contains an arbitrary constant: $y = (x, c)$

▷ one curve in x, y plane for each value of c $\Rightarrow$ family of curves

Example: $y' = -x/y$.

$$\text{separable equation} \Rightarrow \int y \, dy = -\int x \, dx \Rightarrow y^2/2 = -x^2/2 + c$$

$$\text{i.e., } x^2 + y^2 = \text{constant} : \text{ family of circles centered at origin}$$

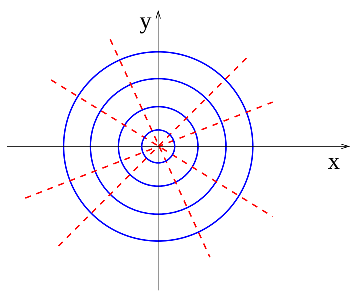

Fig.1

$$\text{Orthogonal family of curves} : \quad y' = -1/f(x, y)$$

$$\text{Example} : \; y' = y/x \; \Rightarrow \; y = cx$$

SUMMARY ON NONLINEAR FIRST-ORDER ODEs

- No general method of solution for 1st-order ODEs beyond linear case; rather, a variety of techniques that work on a case-by-case basis.

Examples:

i) Bring equation to separated-variables form, that is, $y' = \alpha(x)/\beta(y)$;
then equation can be integrated.
Cases covered by this include $y' = \varphi(ax + by)$; $y' = \varphi(y/x)$.

ii) Reduce to linear equation by transformation of variables.
Examples of this include Bernoulli's equation.

iii) Bring equation to exact-differential form, that is
$dy/dx = -M(x, y)/N(x, y)$ such that $M = \partial\phi/\partial x$, $N = \partial\phi/\partial y$.
Then solution determined from $\phi(x, y) =$ const.

SECOND-ORDER LINEAR ODEs

$$f'' + p(x)f' + q(x)f = h(x)$$

- General solution $= \mathrm{PI} + \mathrm{CF}$
  - $\mathrm{CF} = c_1 u_1 + c_2 u_2$

    u_1 and u_2 linearly independent solutions of the homogeneous equation

- 2nd-order linear ODEs with constant coefficients:

$$a_2 f'' + a_1 f' + a_0 f = h(x)$$

▷ Complementary function CF by solving *auxiliary equation*

▷ Particular integral PI by *trial function* with functional form of the inhomogeneous term

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

Complementary function

$$Lf_0 = a_2 \frac{\mathrm{d}^2 f_0}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_0}{\mathrm{d}x} + a_0 f_0 = 0.$$

Try $f_0 = \mathrm{e}^{mx}$ ⇒ $$a_2 m^2 + a_1 m + a_0 = 0.$$

"Auxiliary" equation

$$m_\pm \equiv \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2},$$

$a_1^2 - 4a_2 a_0 \rightarrow +, 0, -$

⇒ Complementary function $$f_0 = A_+ \mathrm{e}^{m_+ x} + A_- \mathrm{e}^{m_- x}.$$

Two constants of integration

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

Complementary function

$$Lf_0 = a_2 \frac{\mathrm{d}^2 f_0}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_0}{\mathrm{d}x} + a_0 f_0 = 0.$$

Try $f_0 = \mathrm{e}^{mx}$ ⇒ $a_2 m^2 + a_1 m + a_0 = 0.$

$$m_\pm \equiv \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2},$$

"Auxiliary" equation

$a_1^2 - 4a_2 a_0 \to +, 0, -$

⇒ Complementary function

$$f_0 = A\mathrm{e}^{mx} + Bx\mathrm{e}^{mx}$$

Repeated roots

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

Complementary function

$$Lf_0 = a_2 \frac{\mathrm{d}^2 f_0}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_0}{\mathrm{d}x} + a_0 f_0 = 0.$$

Particular integral

$$Lf_1 = a_2 \frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_1}{\mathrm{d}x} + a_0 f_1 = h.$$

General solution : $f_0 + f_1$

June 2006

4. Solve the differential equation

$$3\frac{d^2y}{dx^2}+\frac{dy}{dx}-4y=0$$

with initial conditions $y(0)=0$ and $dy/dx(0)=7$.

[Answ.: $3(e^x-e^{-4x/3})$]

September 2007

3. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2}+7\frac{dy}{dt}+12y=3\exp(2t)$$

[Answ.: $c_1e^{-4t}+c_2e^{-3t}+e^{2t}/10$]

June 2008

5. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{-x}$$

[Answ.: $c_1e^{-x}+c_2e^{-2x}+xe^{-x}$]

10. A mass m is suspended on a spring. Its displacement y as a function of time t is described by the equation

September 2009

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\gamma \frac{\mathrm{d}y}{\mathrm{d}t} + \omega_0^2 y = \frac{F_0}{m}\, e^{i\omega t}\,,$$

where γ, ω_0 and F_0 are constants. Initially no driving force is present ($F_0 = 0$). Determine the solution to the equation when $\gamma^2 < \omega_0^2$, subject to the initial conditions $y = y_0$ and $\frac{dy}{dt} = 0$ at $t = 0$.

The driving force is now applied. Find the time dependence of y at sufficiently long times such that all transients have died away. Sketch as a function of ω the amplitude of y and its phase ϕ relative to the driving force.

When $\omega = \omega_0$, show that the average power supplied by the driving force is

$$P_{av} = \frac{F_0^2}{4m\gamma}\,.$$

$$y'' + 2\gamma y' + \omega_0^2 y = (F_0/m)\ e^{i\omega t} \ .$$

CF $(F_0 = 0)$:

Characteristic eq. $\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0 \ \Rightarrow \ \lambda_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

$$\gamma^2 < \omega_0^2 : \lambda_\pm = -\gamma \pm i \underbrace{\sqrt{\omega_0^2 - \gamma^2}}_{\omega'}$$

$$\text{So CF} = e^{-\gamma t}[A \cos \omega' t + B \sin \omega' t] \ .$$

Initial conditions $y(0) = y_0$, $y'(0) = 0$

$\Rightarrow \ A = y_0$, $B\omega' - A\gamma = 0 \ \Rightarrow \ B = y_0 \gamma / \omega'$

$$\text{Thus} \ \ y(t) = y_0 e^{-\gamma t}[\cos \omega' t + (\gamma/\omega') \sin \omega' t] \ .$$

PI (F switched on):

Trial function $y = Ce^{i\omega t} \;\Rightarrow\; C(-\omega^2 + 2\gamma i\omega + \omega_0^2) = F_0/m$

$$\text{So} \;\; C = \frac{F_0/m}{\omega_0^2 - \omega^2 + 2\gamma i\omega} = |C|e^{i\phi}$$

$$\text{where} \;\; |C| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \qquad (amplitude)$$

$$\tan\phi = \frac{2\gamma\omega}{\omega^2 - \omega_0^2} \qquad (phase)$$

$\bullet$*Power P supplied by the driving force F* : $P = F\ \frac{dy}{dt}$

where $F = F_0 e^{i\omega t}$, $y(t) = |C| e^{i(\omega t + \phi)}$, $\frac{dy}{dt} = |C| i\omega e^{i(\omega t + \phi)}$.

- Using the explicit expressions determined previously for amplitude $|C|$ and phase ϕ we get

$$P = \frac{\omega F_0^2 / m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \frac{\sin(2\omega t + \phi) + \sin\phi}{2}$$

$$\text{where} \quad \sin\phi = \frac{\tan\phi}{\sqrt{1 + \tan^2\phi}} = \frac{2\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

- Taking the average power $\langle P \rangle$ ($\langle \sin(2\omega t + \phi) \rangle = 0$)

$$\Rightarrow \langle P \rangle = \frac{\gamma\omega^2 F_0^2 / m}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$$

$$\text{Thus} \quad \langle P \rangle = \frac{F_0^2}{4m\gamma} \quad \text{for} \quad \omega^2 = \omega_0^2 \ .$$

SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

- more than 1 unknown function: $y_1(x), y_2(x), \ldots, y_n(x)$
- set of ODEs that couple $y_1, \ldots, y_n$

▷ physical applications: systems with more than 1 degree of freedom. dynamics couples differential equations for different variables.

Example. System of first-order differential equations:

$$y_1' = F_1(x, y_1, y_2, \ldots, y_n)$$

$$y_2' = F_2(x, y_1, y_2, \ldots, y_n)$$

$$\cdots$$

$$y_n' = F_n(x, y_1, y_2, \ldots, y_n)$$

$\diamondsuit$ Systems of linear ODEs with constant coefficients can be solved by a generalization of the method seen for single ODE:

General solution $= \mathrm{PI} + \mathrm{CF}$

$\triangleright$ Complementary function CF by solving *system of auxiliary equations*

$\triangleright$ Particular integral PI from a *set of trial functions* with functional form as the inhomogeneous terms

June 2007

The variables $\psi(z)$ and $\phi(z)$ obey the simultaneous differential equations

$$3\,\frac{d\phi}{dz} + 5\psi = 2z$$

$$3\,\frac{d\psi}{dz} + 5\phi = 0.$$

Find the general solution for ψ.

- Differentiating 2nd equation wrt z gives

$$0 = 3\,\frac{d^2\psi}{dz^2} + 5\,\frac{d\phi}{dz} = 3\,\frac{d^2\psi}{dz^2} + \frac{5}{3}\,(2z - 5\psi)$$

$$\text{i.e.}\quad \frac{d^2\psi}{dz^2} - \frac{25}{9}\,\psi = -\frac{10}{9}\,z$$

- Solve this 2nd-order linear ODE with constant coefficients:

CF: Auxiliary equation $m^2 - 25/9 = 0 \;\Rightarrow\; m_\pm = \pm 5/3$.
So CF $= Ae^{5z/3} + Be^{-5z/3}$.

PI: Trial function $\psi_0 = C_1 z + C_0 \;\Rightarrow$
$-(25/9)C_1 z - (25/9)C_0 = -10/9z$
$\Rightarrow\; C_1 = 2/5\;,\; C_0 = 0 \;\Rightarrow\; \mathrm{PI} = 2z/5$.

General solution for ψ :

$$\psi(z) = \mathrm{CF} + \mathrm{PI} = A\; e^{5z/3} + B\; e^{-5z/3} + \frac{2}{5}\; z\;.$$

June 2008

10. Consider the coupled differential equations

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} + au - bv &= f \\ \frac{\mathrm{d}v}{\mathrm{d}t} + av + bu &= 0\end{aligned}$$

where a, b and f are constants.

i) Solve them for $f = 0$, subject to the boundary conditions $u = 0$ and $v = v_0$ when $t = 0$. [10]

ii) Solve them for $f \neq 0$, subject to the boundary conditions $u = v = 0$ when $t = 0$, and write down the steady state solutions. [10]

- Using notation $D = d/dt$,

$$(D + a)u - bv = f$$

$$bu + (D + a)v = 0$$

- Apply $D + a$ on 1st eq. and multiply 2nd eq. by b:

$$(D + a)^2 u - b(D + a)v = (D + a)f$$

$$b^2 u + b(D + a)v = 0$$

- Add the two equations:

$$[(D + a)^2 + b^2]u = af$$

which is 2nd-order ODE with constant coefficients

$$u'' + 2au' + (b^2 + a^2)u = af \quad .$$

General solution

CF: Auxiliary equation $\lambda^2 + 2a\lambda + (b^2 + a^2) = 0 \;\Rightarrow\; \lambda_\pm = -a \pm ib$.

So $\mathrm{CF} = e^{-at}[A\cos bt + B\sin bt]$.

PI: Trial function: constant $u_0 = C \;\Rightarrow\; C = af/(a^2 + b^2)$.

General solution for u :

$$u(t) = \mathrm{CF} + \mathrm{PI} = e^{-at}[A\cos bt + B\sin bt] + af/(a^2 + b^2)\ .$$

General solution for v :

$$v(t) = b^{-1}[(D + a)u - f] = e^{-at}[-A\sin bt + B\cos bt] - bf/(a^2 + b^2)\ .$$

i) $f = 0, u(0) = 0$ and $v(0) = v_0 \Rightarrow A = 0, B = v_0$

$$\text{Therefore} \quad u(t) = v_0 e^{-at} \sin bt \quad ,$$

$$v(t) = v_0 e^{-at} \cos bt \quad .$$

ii) $f \neq 0, u(0) = v(0) = 0 \Rightarrow A = -af/(a^2 + b^2), B = bf/(a^2 + b^2)$

$$\text{Therefore} \quad u(t) = f/(a^2 + b^2)[e^{-at}(-a \cos bt + b \sin bt) + a] \quad ,$$

$$v(t) = f/(a^2 + b^2)[e^{-at}(a \sin bt + b \cos bt) - b] \quad .$$

Steady state solutions $t \to \infty$:

$$u = af/(a^2 + b^2) \; , \quad v = -bf/(a^2 + b^2) \; .$$