Revision Lectures

Trinity Term 2012

Lecturer: F Hautmann

CP3

- Complex Numbers and Ordinary Differential Equations

Lecture times:

Monday week 2 at 11 am

Tuesday week 2 at 11 am

Venue: Martin Wood Lecture Theatre

- Slides will be posted on lecture webpage: http://www-thphys.physics.ox.ac.uk/people/FrancescoHautmann/Cp4rev/

I. Complex numbers

1 Basic algebra and geometry of the complex plane

2 Elementary functions of complex variable

3 De Moivre's theorem and applications

4 Curves in the complex plane

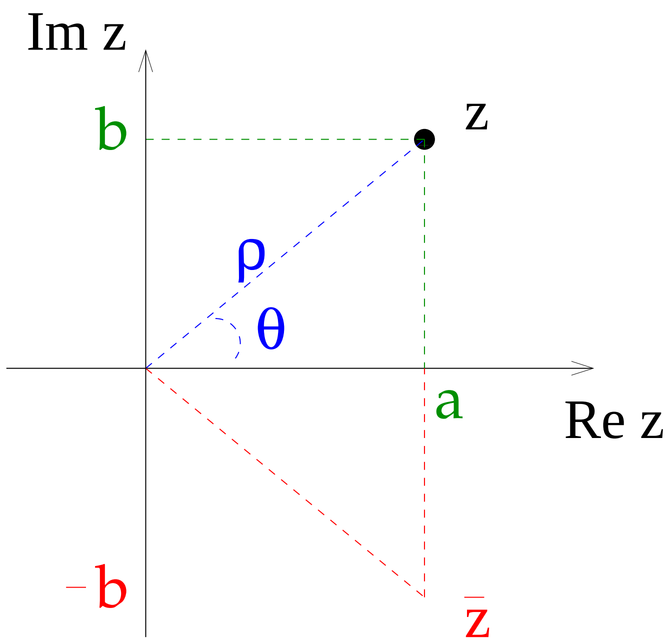
5 Roots of complex numbers and polynomials

1. THE COMPLEX PLANE

◊ $\mathbb{C}$ = set of complex nos with $+$, $\times$ operations

$$z = a + ib \quad , \qquad i^2 = -1$$

$$a = \mathrm{Re} z \quad , \qquad b = \mathrm{Im} z$$



ρ = mod(z) = | z |

θ = arg(z)

$$z = a + ib = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta)$$
$$\overline{z} = a - ib = \rho e^{-i\theta} = \rho(\cos\theta - i\sin\theta)$$

2. Elementary functions on $\mathbb{C}$

- Complex polynomials and rational functions defined by algebraic operations in $\mathbb{C}$

- Complex exponential: $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

$\longrightarrow$ complex trigon. and hyperb. fctns in terms of exp.

e.g. $\sin z = (e^{iz} - e^{-iz})/(2i)$

$\sinh z = (e^z - e^{-z})/2$

- Complex logarithm $\ln z$: $e^{\ln z} = z$

$\Rightarrow \ln z = \ln |z| + i(\theta + 2n\pi)$, $n = 0, \pm 1, \ldots$ ($\leftarrow$ multi-valued)

$\longrightarrow$ complex powers: $z^\alpha = e^{\alpha \ln z}$ (α complex)

3. DE MOIVRE'S THEOREM

de Moivre's theorem and trigonometric identities

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

For r=1

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

De Moivre

e.g. n=2 :

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\cos\theta\sin\theta$$

June 2006

3. Express $\tan 3\theta$ in terms of $\tan\theta$.

$$\text{De Moivre} \Rightarrow \cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\text{Then : } \tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

June 2008

1. Prove that

$$\sin^5\theta = \frac{1}{2^4}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

$$\sin^5\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^5 = [e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}]/(i\ 2^5)$$

$$= [\sin 5\theta - 5\sin 3\theta + 10\sin\theta]/2^4$$

June 2008

9.a) Prove the identity

$$\sum_{n=0}^{N-1} \cos(nx) = \frac{\sin(Nx/2)}{\sin(x/2)} \; \cos((N-1)x/2) \; .$$

$$\sum_{n=0}^{N-1} \cos(nx) = \text{Re} \sum_{n=0}^{N-1} e^{inx} = \text{Re} \; \Big(\sum_{n=0}^{\infty} e^{inx} - \underbrace{\sum_{n=N}^{\infty} e^{inx}}_{e^{iNx} \sum_{n=0}^{\infty} e^{inx}} \Big)$$

$$= \text{Re} \left[\left(1 - e^{iNx}\right) \sum_{n=0}^{\infty} e^{inx} \right] = \text{Re} \; \frac{1 - e^{iNx}}{1 - e^{ix}}$$

$$= \text{Re} \; \frac{e^{iNx/2}[-2i \sin(Nx/2)]}{e^{ix/2}[-2i \sin(x/2)]} = \frac{\sin(Nx/2)}{\sin(x/2)} \; \text{Re} \; e^{i(N-1)x/2}$$

$$= \frac{\sin(Nx/2)}{\sin(x/2)} \; \cos((N-1)x/2)$$

The complex logarithm

$\ln z$

$$e^{\ln z} = z \quad = |z|\, e^{\imath\theta} = e^{\ln|z|} e^{\imath\theta} = e^{\ln|z| + \imath\theta}$$

$$\Rightarrow \quad \boxed{\ln z = \ln|z| + \imath \arg(z)}$$

Need to know θ including 2πn phase ambiguity in z

$$\ln z = \ln|z| + i(\theta + 2\pi n) \quad , \qquad n \text{ integer}$$

- different $n \Rightarrow$ different "branches" of the logarithm
- $n = 0$: "principal" branch
- $\ln z \rightarrow$ example of "multi-valued" function

Note: It follows that the formula

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

is valid, once the branch of $\ln z_1$ and $\ln z_2$ is chosen, only for a given choice of the branch for $\ln(z_1 z_2)$. In general,

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2 + 2\pi i k \quad \text{for some } k \in \mathbb{Z}$$

June 2011 . 1. Prove the above formula.

Proof. Let $\ln z_1 = w_1$, $\ln z_2 = w_2$, and $\ln(z_1 z_2) = w_3$. Then

$$e^{w_3} = e^{\ln(z_1 z_2)} = z_1 z_2 = e^{w_1} e^{w_2} = e^{w_1 + w_2}$$

$$\text{which means } \; w_3 = w_1 + w_2 + 2\pi i k \; \text{ for some } k \in \mathbb{Z} \; ,$$

$$\text{that is} \, , \;\; \ln(z_1 z_2) = \ln z_1 + \ln z_2 + 2\pi i k \; .$$

Example: evaluation of $\ln(-1)$

$$\ln(-1) = \ln e^{i\pi} = \ln 1 + i(\pi + 2\pi n) = i\pi + 2\pi i n \ , \quad n \ \text{integer}$$

$$\ln(-1) = i\pi \text{ for } n = 0 \text{ (principal branch)}$$

Note: with $z_1 = -1$, $z_2 = -1$, the formula

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

is satisfied, once the principal branch value $\ln(-1) = i\pi$ is taken, only for the branch in which

$$\ln(-1)^2 = \ln 1 = 2\pi i \ .$$

<u>June 2007</u>

Find all possible values of $\ln(z) - \ln(-z)$
where z is a complex number.

$$z = |z|e^{i\theta}$$

$$\ln z = \ln|z| + i(\theta + 2\pi n) \quad , \qquad n \ \text{integer}$$

$$\ln(-z) = \ln|z| + i(\pi + \theta + 2\pi m) \quad , \qquad m \ \text{integer}$$

$$\Rightarrow \ \ln(z) - \ln(-z) = -i\pi + 2i(\underbrace{n - m}_{k})\pi = i(2k-1)\pi$$

4. CURVES IN THE COMPLEX PLANE

• locus of points satisfying constraint in complex variable z

♠ Curves can be defined

▷ either directly by constraint in complex variable z

$$\text{ex.}: \quad |z| = R$$

▷ or by parametric form $\gamma(t) = x(t) + iy(t)$

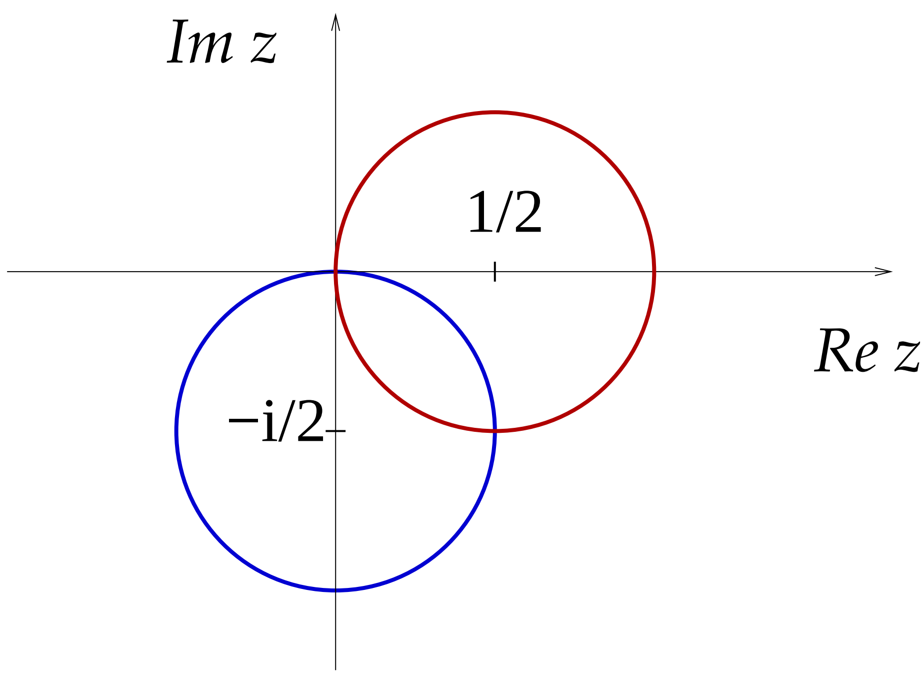
$$\text{ex.}: \quad x(t) = R\cos t$$

$$y(t) = R\sin t\ , \quad 0 \leq t < 2\pi$$

September 2009

Sketch the loci in the complex plane representing the conditions

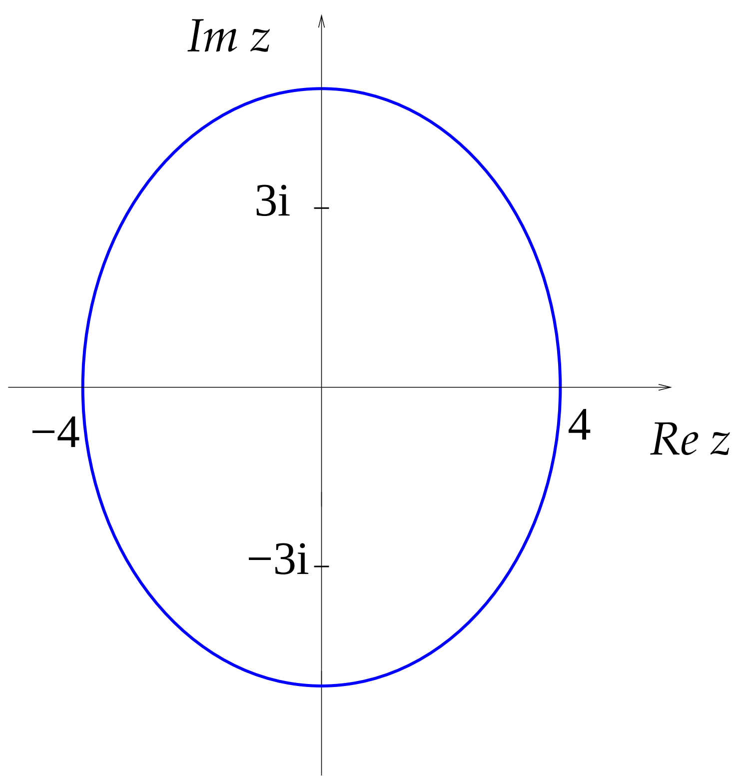
$$Re(z^{-1}) = 1 \text{ and } Im(z^{-1}) = 1\,.$$



<u>September 2011</u>

Draw the set of points in the complex plane such that

$$|z - 3i| + |z + 3i| = 10 \ .$$

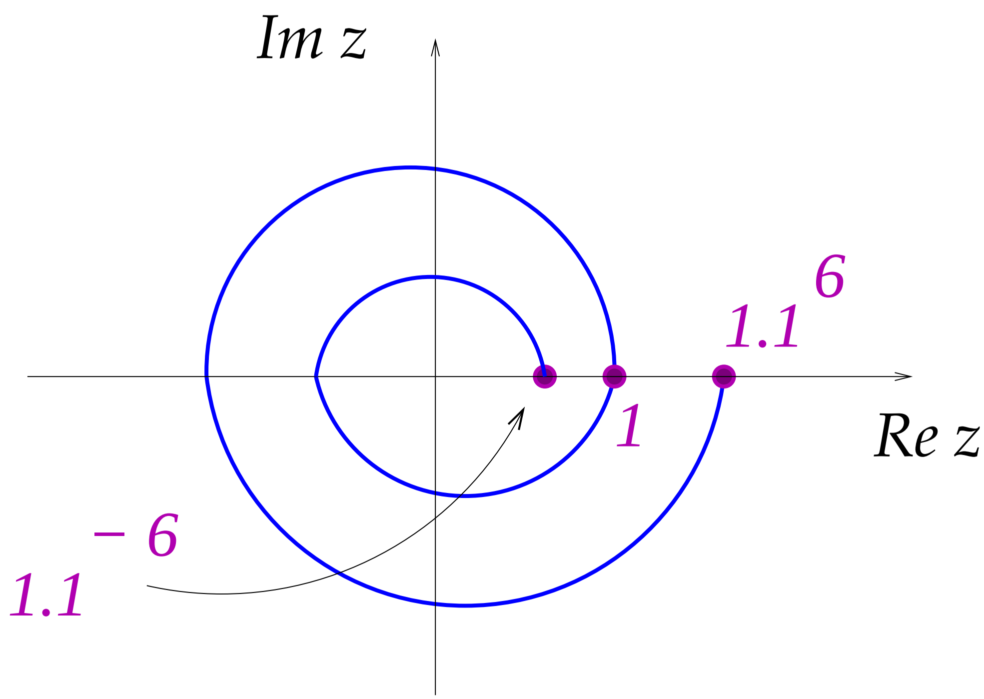


June 2007

Sketch the curve in the complex plane

$$z = a^t e^{ibt} \quad , \qquad -6 \leq t \leq 6$$

where a and b are real constants given by $a = 1.1$, $b = \pi/3$.



5. ROOTS OF COMPLEX NUMBERS

Def.:

- A number u is said to be an n-th root of complex number z if $u^n = z$, and we write $u = z^{1/n}$.

Th.:

- Every complex number has exactly n distinct n-th roots.

Let $z = r(\cos\theta + i\sin\theta)$; $u = \rho(\cos\alpha + i\sin\alpha)$. Then

$$r(\cos\theta + i\sin\theta) = \rho^n(\cos\alpha + i\sin\alpha)^n = \rho^n(\cos n\alpha + i\sin n\alpha)$$

$$\Rightarrow \; \rho^n = r \;\; , \quad n\alpha = \theta + 2\pi k \qquad (k \text{ integer})$$

$$\text{Thus } \; \rho = r^{1/n} \;\; , \quad \alpha = \theta/n + 2\pi k/n \; .$$

n distinct values for k from 0 to $n-1$. $(z \neq 0)$

$$\text{So } \; u = z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right] \;\; , \quad k = 0, 1, \ldots, n-1$$

Roots of polynomials

$$P(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 .$$

$$P(z = z_i) = 0 \qquad \Rightarrow z_i \text{ is a root}$$

Characterising a polynomial by its roots

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n (z - z_1)(z - z_2) \cdots (z - z_n)$$

$$= a_n \left(z^n - z^{n-1} \sum_{j=1}^{n} z_j + \cdots + (-1)^n \prod_{j=1}^{n} z_j \right).$$

Comparing coefficients of z^{n-1} and z^0

$$\frac{a_{n-1}}{a_n} = -\sum_j z_j \quad ; \quad \frac{a_0}{a_n} = (-1)^n \prod_j z_j$$

<u>June 2011</u> **2**. Determine the real numbers x and y which satisfy

$$\sum_{k=0}^{100} e^{i\pi k/3} = x + iy \ .$$

Observe that $e^{i\pi k/3} = e^{2i\pi k/6}$ are the sixth roots of unity for $k = 0, 1, \ldots, 5$

$$\Rightarrow \sum_{k=0}^{5} e^{i\pi k/3} = 0$$

The same applies for subsequent sets of 6 values of k.
Thus the first $6 \times 16 = 96$ terms of the sum give zero, and the last five must equal minus the root with $k = 5$, so that

$$\sum_{k=0}^{100} e^{i\pi k/3} = \sum_{k=96}^{100} e^{i\pi k/3} = -e^{i\pi 5/3} = e^{i\pi 2/3} = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2} \ \Rightarrow\ x = -\frac{1}{2} \ , \quad y = \frac{\sqrt{3}}{2}$$

$$\text{Alternatively :} \quad \sum_{k=0}^{100} e^{i\pi k/3} = \sum_{k=0}^{\infty} e^{i\pi k/3} - \underbrace{\sum_{k=101}^{\infty} e^{i\pi k/3}}_{e^{i\pi 101/3} \sum_{k=0}^{\infty} e^{i\pi k/3}} = \left(1 - e^{i\pi 101/3}\right) \sum_{k=0}^{\infty} e^{i\pi k/3}$$

$$= \frac{1 - e^{i5\pi/3}}{1 - e^{i\pi/3}} = \frac{e^{i5\pi/6} \sin(5\pi/6)}{e^{i\pi/6} \sin(\pi/6)} = e^{i\pi 2/3} = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}$$

June 2008

9.b) Find all the solutions of the equation

$$\left(\frac{z+i}{z-i}\right)^n = -1\,,$$

and solve $z^4 - 10z^2 + 5 = 0$.

$$\left(\frac{z+i}{z-i}\right)^n = -1 = e^{i(\pi+2N\pi)} \quad , \qquad N \text{ integer}$$

$$\Rightarrow \qquad \frac{z+i}{z-i} = e^{i(\pi/n+2N\pi/n)} \quad , \qquad N = 0, 1, \ldots, n-1$$

Then : $z = i \, \dfrac{e^{i(\pi/n+2N\pi/n)}+1}{e^{i(\pi/n+2N\pi/n)}-1} = i \, \dfrac{\cos[\pi(1+2N)/(2n)]}{i \, \sin[\pi(1+2N)/(2n)]} = \text{cotg} \, \dfrac{\pi(1+2N)}{2n}$.

For $n = 5$: $(z+i)^5 = -(z-i)^5 \;\Rightarrow\; z(z^4-10z^2+5) = 0$. Then the 4 roots of $z^4-10z^2+5 = 0$

are $\quad \text{cotg} \, \dfrac{\pi}{10} \,, \text{cotg} \, \dfrac{3\pi}{10} \,, \text{cotg} \, \dfrac{7\pi}{10} \,, \text{cotg} \, \dfrac{9\pi}{10}$.

10. (a) State de Moivre's theorem. [2]

(b) Use de Moivre's theorem to show that

$$\cos 6\theta = 4\cos^3 2\theta - 3\cos 2\theta.$$

Hence find all the solutions to the cubic equation

$$4x^3 - 3x = \frac{\sqrt{3}}{2}.$$

[9]

(c) Solve the equation

$$\left(\frac{z+1}{z-1}\right)^n = i$$

for n an integer. Hence solve the equation

$$z^3 + 3iz^2 + 3z + i = 0.$$

[9]

<u>September 2010</u>

a) De Moivre : $(\cos\theta + i\sin\theta)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

b) $\cos 6\theta + i\sin 6\theta = e^{i6\theta} = (e^{i2\theta})^3 = (\cos 2\theta + i\sin 2\theta)^3$

$$\Rightarrow \ \cos 6\theta = \cos^3 2\theta - 3\cos 2\theta \sin^2 2\theta = 4\cos^3 2\theta - 3\cos 2\theta$$

$$4x^3 - 3x = \frac{\sqrt{3}}{2}$$

Then : $x = \cos 2\theta \ \Rightarrow \ \cos 6\theta = \frac{\sqrt{3}}{2}$

$$6\theta = \pi/6 \Rightarrow \ x = \cos 2\theta = \cos(\pi/18)$$

$$6\theta = 11\pi/6 \Rightarrow \ x = \cos 2\theta = \cos(11\pi/18)$$

$$6\theta = 13\pi/6 \Rightarrow \ x = \cos 2\theta = \cos(13\pi/18)$$

c) $\left(\frac{z+1}{z-1}\right)^n = i \ \Rightarrow \ \frac{z+1}{z-1} = e^{i(\pi/(2n) + 2N\pi/n)} \ , \quad N = 0, 1, \ldots, n-1$

Then $z = \frac{e^{i(\pi/(2n)+2N\pi/n)} + 1}{e^{i(\pi/(2n)+2N\pi/n)} - 1} = \frac{\cos[\pi(1+4N)/(4n)]}{i \ \sin[\pi(1+4N)/(4n)]} = -i \ \text{cotg} \ \frac{\pi(1+4N)}{4n}$

For $n = 3$: $(z+1)^3 = i(z-1)^3 \ \Rightarrow \ z^3 + 3iz^2 + 3z + i = 0$. Then the 3 roots of $z^3 + 3iz^2 + 3z + i = 0$ are $-i \ \text{cotg} \ \frac{\pi}{12} \ , -i \ \text{cotg} \ \frac{5\pi}{12} \ , -i \ \text{cotg} \ \frac{3\pi}{4}$.

September 2003

2. Solve the equation

$$z^6 + z^3 + 1 = 0.$$

$$u = z^3, \quad u^2 + u + 1 = 0, \quad u = \frac{-1 \pm \sqrt{3}i}{2} = e^{i(2\pi/3 + n\pi)}, \quad n = 0,1$$

$$z = u^{1/3} = e^{i(2\pi/9 + n\pi/3 + 2m\pi/3)}, \quad n = 0,1, \quad m = 0,1,2$$

$2b$. Solve the equation $z^6 - 15z^4 + 15z^2 - 1 = 0$

$(x+y)^0$						1					
$(x+y)^1$					1		1				
$(x+y)^2$				1		2		1			
$(x+y)^3$			1		3		3		1		
$(x+y)^4$		1		4		6		4		1	
$(x+y)^5$	1		5		10		10		5		1

$$(z+i)^6 = z^6 + 6iz^5 - 15z^4 - 20iz^3 + 15z^2 + 6iz - 1$$

$$(z+i)^6 + (z-i)^6 = 2\left(z^6 - 15z^4 + 15z^2 - 1\right) = 0 \quad \Rightarrow \quad \frac{(z+i)^6}{(z-i)^6} \equiv u^6 = -1 = e^{i\pi}$$

$$u = e^{i(\pi/6 + 2\pi n/6)}, \quad n = 0,1,2,3,4,5$$

$$z - i = u(z+i) \quad \Rightarrow \quad z = \frac{i(1+u)}{(1-u)}$$