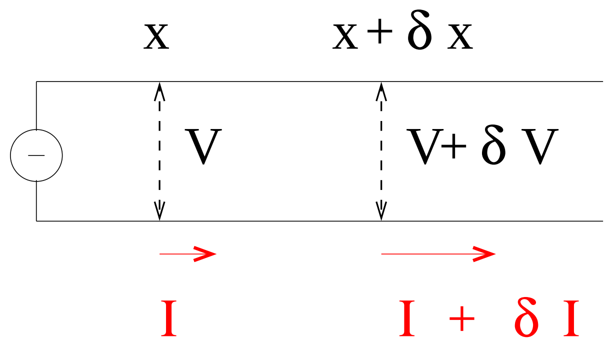
Lecture 8 — Waves

Final examples of wave motion

◊ Waves on electrical lines

◊ Longitudinal elastic waves

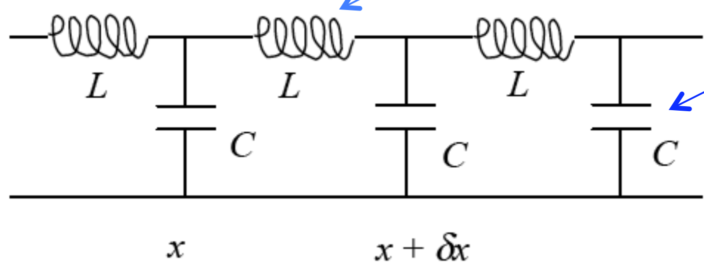
Transmission lines



voltage and current signals propagate along the line

- self-inductance of the element $L = L' \ \delta x$
- electric capacitance of the element $C = C' \ \delta x$
- ideal line $=$ lossless $\Rightarrow$ zero resistance

Waves on electrical lines



$$L\frac{\partial I}{\partial t} = -\delta V = -\frac{\partial V}{\partial x}\delta x$$

$$\frac{\partial Q}{\partial t} = C\frac{\partial V}{\partial t} = -\delta I = -\frac{\partial I}{\partial x}\delta x$$

$$\frac{L}{\delta x}\frac{\partial I}{\partial t} = L'\frac{\partial I}{\partial t} = -\frac{\partial V}{\partial x}$$

$$\frac{C}{\delta x}\frac{\partial V}{\partial t} = C'\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}$$

$$\frac{\partial^2 V}{\partial t \partial x} = -\frac{1}{C'}\frac{\partial^2 I}{\partial x^2} = -L'\frac{\partial^2 I}{\partial t^2}$$

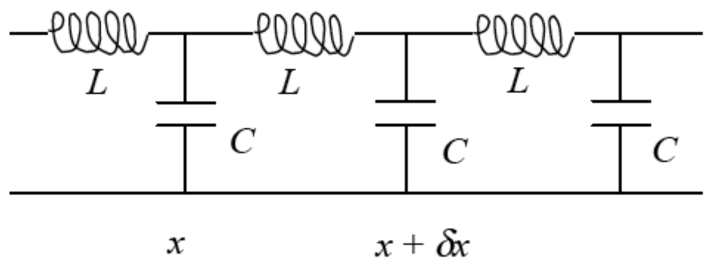
$$\frac{\partial^2 I}{\partial x \partial t} = -C'\frac{\partial^2 V}{\partial t^2} = -\frac{1}{L'}\frac{\partial V}{\partial x^2}$$

$$\boxed{\frac{\partial^2 I}{\partial x^2} = L'C'\frac{\partial^2 I}{\partial t^2}}$$

$$\mathrm{v} = 1/\sqrt{L'C'}$$

$$\boxed{\frac{\partial^2 V}{\partial x^2} = L'C'\frac{\partial^2 V}{\partial t^2}}$$

Waves on electrical lines



$$L\frac{\partial I}{\partial t} = -\delta V = -\frac{\partial V}{\partial x}\delta x$$

$$\frac{\partial Q}{\partial t} = C\frac{\partial V}{\partial t} = -\delta I = -\frac{\partial I}{\partial x}\delta x$$

$$\frac{L}{\delta x}\frac{\partial I}{\partial t} = L'\frac{\partial I}{\partial t} = -\frac{\partial V}{\partial x}$$

$$\frac{C}{\delta x}\frac{\partial V}{\partial t} = C'\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}$$

Impedance

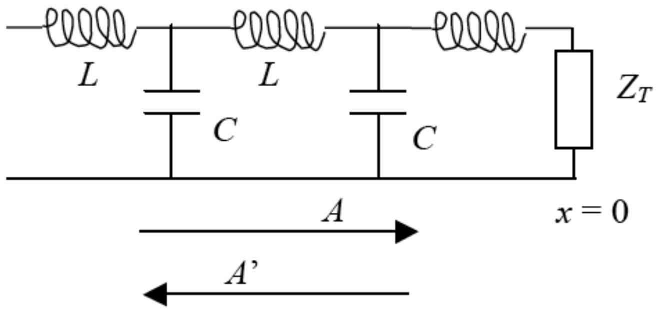
$$V = V_0 \sin\left(\omega t \mp kx\right) = IZ_{\mp} \qquad I = \left(V_0 / Z\right)\sin\left(\omega t \mp kx\right)$$

$$\frac{\partial I}{\partial t} = -\frac{1}{L'}\frac{\partial V}{\partial x} \Rightarrow \frac{V_0}{Z_{\mp}}\omega = \frac{V_0}{L'}k$$

$$\boxed{Z_{\mp} = \pm\sqrt{L'/C'}}$$

$$\mathrm{v} = \frac{\omega}{k} = 1/\sqrt{L'C'}$$

Reflection at a terminated line



$$Z_{\mp} = \pm\sqrt{L'/C'}$$

$$V = A\exp\left(i\left(\omega t - kx\right)\right) + A'\exp\left(i\left(\omega t + kx\right)\right)$$

$$Z\_I = A\exp\left(i\left(\omega t - kx\right)\right) - A'\exp\left(i\left(\omega t + kx\right)\right)$$

$$\frac{V(x=0)}{I(x=0)} = Z_T$$

$$\frac{V}{Z\_I}\Big|_{x=0} = \frac{Z_T}{Z\_} = \frac{A + A'}{A - A'}$$

$$r = \frac{A'}{A} = \frac{Z_T - Z\_}{Z_T + Z\_}$$

Hence

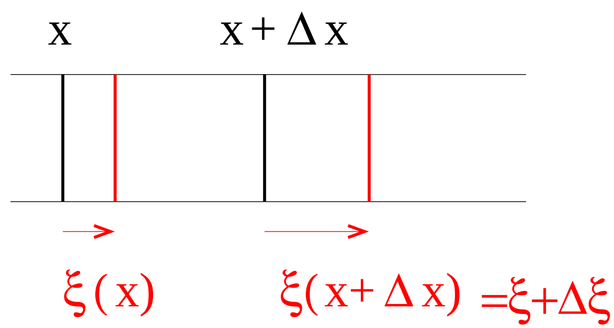
when $Z_T \to 0$, $r \to -1$

when $Z_T = Z_0$, $r = 0$

when $Z_T \to \infty$, $r \to +1$

LONGITUDINAL ELASTIC WAVES

Longitudinal oscillations of a rod



$$\text{strain} = \frac{\text{elongation}}{\text{original length}} = \frac{\Delta\xi}{\Delta x} \longrightarrow \frac{\partial\xi}{\partial x}$$

$$\text{stress} = \frac{\text{force}}{\text{area}} = \frac{F}{A}$$

For small strain: stress = Y (strain) , Y Young modulus

$$\Rightarrow \underbrace{\rho\ A\ \Delta x}_{\Delta m}\ \frac{\partial^2 \xi}{\partial t^2} = A\ Y \underbrace{[\frac{\partial \xi}{\partial x}(x + \Delta x) - \frac{\partial \xi}{\partial x}(x)]}_{\frac{\partial^2 \xi}{\partial x^2}\ \Delta x}$$

$$\text{Thus} \quad \frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\ \frac{\partial^2 \xi}{\partial t^2} \quad , \quad v = \sqrt{Y/\rho}$$

Example

Take rod of length L with one fixed end and one free end:

$$\xi(0,t) = 0 \qquad \partial\xi/\partial x(L,t) = 0$$

- Determine the normal frequencies. What is the lowest normal mode frequency?

— — — —

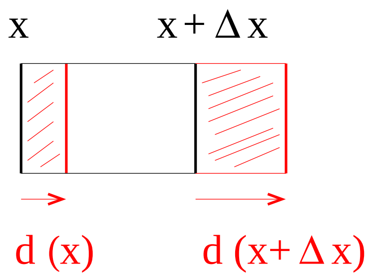
$$\xi(x,t) = \left(A \sin\frac{\omega x}{v} + B \cos\frac{\omega x}{v}\right)\cos(\omega t + \phi)$$

$$\xi(0,t) = 0 \;\Rightarrow\; B = 0 \quad ; \qquad \frac{\partial\xi}{\partial x}(L,t) = 0 \;\Rightarrow\; \cos\frac{\omega L}{v} = 0 \;\Rightarrow\; \frac{\omega L}{v} = \frac{\pi}{2} + n\pi$$

- Thus the normal frequencies are

$$\omega_n = \frac{\pi v}{L}(n + \frac{1}{2}) = \frac{\pi}{L}\sqrt{\frac{Y}{\rho}}(n + \frac{1}{2}) \qquad (n \;\; \text{integer})$$

$$\text{lowest normal mode frequency} : \qquad \omega_0 = \frac{\pi}{2L}\sqrt{\frac{Y}{\rho}}$$



$$\Delta V = A[d(x+\Delta x) - d(x)] = A\frac{\partial d}{\partial x}\,\Delta x = V\frac{\partial d}{\partial x}$$

$$\underbrace{\rho\ A\ \Delta x}_{\Delta m}\ \frac{\partial^2 d}{\partial t^2} = A[\underbrace{p(x) - p(x+\Delta x)}_{-(\partial p/\partial x)\ \Delta x}]\ ,\quad \text{i.e.,}\ \ \rho\ddot{d} = -\partial p/\partial x \quad \text{(Euler equation)}$$

- Variation of pressure with volume is controlled by bulk modulus K:

$$K = -\frac{\partial p}{\partial \ln V} = -V\ \frac{\partial p}{\partial V}$$

$$\Rightarrow \quad p = p_0 + \frac{\partial p}{\partial V}\ \Delta V = p_0 + \left(-\frac{K}{V}\right)\ \left(V\ \frac{\partial d}{\partial x}\right) = p_0 - K\ \frac{\partial d}{\partial x}$$

$$\text{Therefore} \quad \frac{\partial p}{\partial x} = -K\ \frac{\partial^2 d}{\partial x^2}$$

- Plugging this into Euler equation yields the equation of sound waves in a compressible gas:

$$\frac{\partial^2 d}{\partial x^2} = \frac{1}{v^2}\ \frac{\partial^2 d}{\partial t^2}\quad ,\quad v = \sqrt{K/\rho}$$

$$\frac{\partial^2 d}{\partial x^2} = \frac{\rho}{K}\frac{\partial^2 d}{\partial t^2}$$

- Isothermal compressions: $PV = \text{constant} \Rightarrow K = -V\frac{\partial P}{\partial V} = P \Rightarrow \text{v} = \sqrt{P/\rho}$
- Adiabatic compressions: $PV^\gamma = \text{constant} \Rightarrow K = -V\frac{\partial P}{\partial V} = \gamma P \Rightarrow \text{v} = \sqrt{\gamma P/\rho}$

From kinetic theory we know $P = \frac{1}{3}\rho\overline{\text{v}^2}$ where here v is the molecular speed. Hence $\text{v}_{sound} = \sqrt{\frac{\gamma}{3}\overline{\text{v}^2}}$ and thus $\text{v}_{sound} \approx \text{v}_{rms}$ of molecules since sound is transmitted by moving molecules.

Characteristic impedance $$Z = \frac{K\frac{\partial d}{\partial x}}{\frac{\partial d}{\partial t}} \quad \left(= \frac{P}{\text{v}}\right)$$

Wave in x-direction $$Z = \frac{Kk}{\omega} = \frac{K}{v} = (\rho K)^{1/2}$$

Sound waves in a 3-dimensional compressible gas

▷ sound waves propagate in all three directions

▷ displacements d_i, $i = 1, 2, 3$, for each point $\mathbf{x} = (x, y, z)$:

$\mathbf{d} = (d_1, d_2, d_3)$ "displacement field"

▷ each d_i satisfies Euler equation of the form found in 1-dimensional case:

$$\rho \, \ddot{d}_i = -\partial p / \partial x_i \quad , \qquad \text{i.e.,} \quad \rho \, \ddot{\mathbf{d}} = - \text{ grad } p$$

• Variation of volume in 3-d is

$$\Delta V = \Delta y \Delta z [d_1(x + \Delta x, y, z) - d_1(x, y, z)] + \Delta z \Delta x [d_2(x, y + \Delta y, z) - d_2(x, y, z)]$$
$$+ \Delta x \Delta y [d_3(x, y, z + \Delta z) - d_3(x, y, z)] = \Delta x \Delta y \Delta z \left(\frac{\partial d_1}{\partial x} + \frac{\partial d_2}{\partial y} + \frac{\partial d_3}{\partial z} \right) = V \text{ div } \mathbf{d}$$

• Therefore $\quad p = p_0 + \dfrac{\partial p}{\partial V} \, \Delta V = p_0 + \left(-\dfrac{K}{V} \right) \, (V \text{ div } \mathbf{d}) = p_0 - K \text{ div } \mathbf{d}$

$$\Rightarrow \quad \frac{\partial p}{\partial x_i} = -K \, \frac{\partial}{\partial x_i} \text{ div } \mathbf{d} \quad , \qquad \text{i.e.,} \quad \text{grad } p = -K \text{ grad div } \mathbf{d}$$

• Thus $\quad \rho \, \ddot{\mathbf{d}} = K \text{ grad div } \mathbf{d}$

- Sound waves in 3-dimensional compressible gas:

$$\rho \, \ddot{\mathbf{d}} = K \text{ grad div } \mathbf{d}$$

- Displacement field in terms of scalar potential φ

$$\mathbf{d} = \text{grad } \varphi$$

$\Rightarrow$ wave equation in φ

$$\nabla^2 \varphi - \frac{1}{v^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad , \qquad v = \sqrt{K/\rho}$$

longitudinal pressure waves

$$\nabla^2 = \text{div grad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$