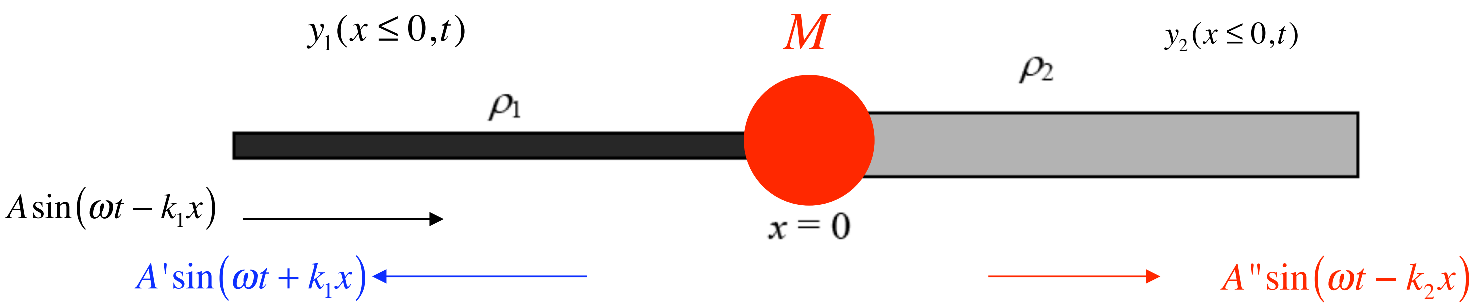
Effects of boundaries on wave propagation

- ♢ Inhomogeneous string
  (see last lecture)

- ♢ Pointlike mass at the boundary

- ♢ Characteristic impedance

Reflection from a mass at the boundary

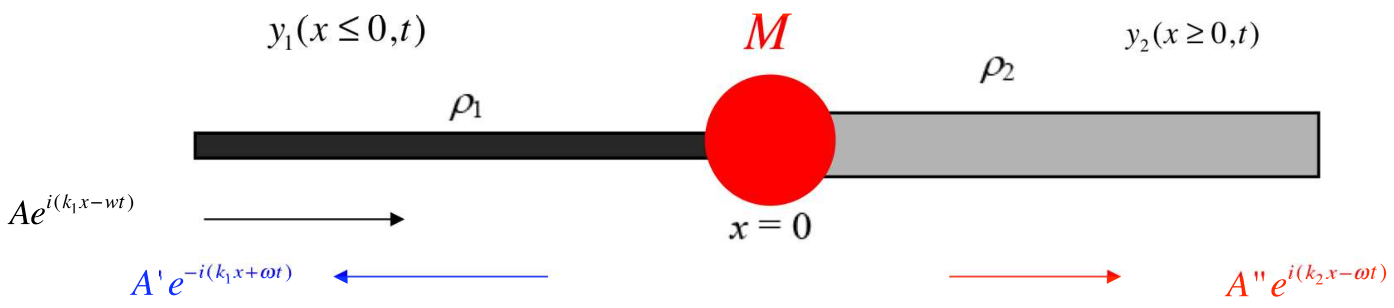


To keep track of phase changes easier to use complex exponentials

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 y}{\partial t^2} \qquad y(x,t) = Be^{i(kx-\omega t)} \text{ is a solution}$$

$$\left(y(x,t) = \mathrm{Re}(Be^{i(kx-\omega t)}) \text{ or } \mathrm{Im}(Be^{i(kx-\omega t)}) \text{ are also solutions} \right)$$

Reflection from a mass at the boundary



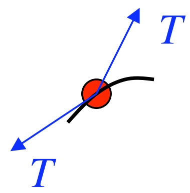
$$y_1(x,t) = Ae^{i(k_1 x - wt)} + A' e^{-i(k_1 x + \omega t)}$$

$$y_2(x,t) = A'' e^{i(k_2 x - \omega t)}$$

$$y_1(x = 0-, t) = y_2(x = 0+, t)$$

String continuous

$$A + A' = A''$$



$$-T \frac{\partial y_1}{\partial x}\Big|_{x=0} + T \frac{\partial y_2}{\partial x}\Big|_{x=0} = M \frac{\partial^2 y_1}{\partial t^2}\Big|_{x=0} = M \frac{\partial^2 y_2}{\partial t^2}\Big|_{x=0}$$

Newton's 2nd law

$$-ik_1 TA + ik_1 TA' + ik_2 TA'' = -\omega^2 M\left(A + A'\right) = -\omega^2 MA'' \quad \Rightarrow \quad ik_1\left(A - A'\right) = \left(ik_2 + \omega^2 M / T\right) A''$$

$$A + A' = A''$$

$$ik_1\left(A - A'\right) = \left(ik_2 + \omega^2 M / T\right) A''$$

$$r = \frac{A'}{A} = \frac{\left(k_1 - k_2\right)T + i\omega^2 M}{\left(k_1 + k_2\right)T - i\omega^2 M} = \mathrm{R}e^{i\theta}$$

$$t = \frac{A''}{A} = \frac{2k_1 T}{\left(k_1 + k_2\right)T - i\omega^2 M} = Se^{i\phi}$$

$$R = \left[\frac{\left(k_1 - k_2\right)^2 T^2 + \omega^4 M^2}{\left(k_1 + k_2\right)^2 T^2 + \omega^4 M^2}\right]^{1/2}, \quad \theta = \tan^{-1}\left(\frac{\omega^2 M}{k_1 - k_2}\right) + \tan^{-1}\left(\frac{\omega^2 M}{k_1 + k_2}\right)$$

$$S = \left[\frac{4k_1^2 T^2}{\left(k_1 + k_2\right)^2 T^2 + \omega^4 M^2}\right]^{1/2}, \quad \phi = \tan^{-1}\left(\frac{\omega^2 M}{k_1 + k_2}\right)$$

$$M = 0$$

$$\frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

$$\frac{A''}{A} = \frac{2k_1}{k_1 + k_2}$$

$$k_1 < k_2 \ \left(\rho_1 < \rho_2 \dagger\right):$$

$$\theta = \pi$$

$$\phi = 0$$

$\dagger$ $k = \omega\sqrt{\frac{\rho}{T}}$

$$A + A' = A''$$

$$ik_1\left(A - A'\right) = \left(ik_2 + \omega^2 M / T\right) A''$$

$$r = \frac{A'}{A} = \frac{\left(k_1 - k_2\right)T + i\omega^2 M}{\left(k_1 + k_2\right)T - i\omega^2 M} = \mathrm{Re}^{i\theta}$$

$$t = \frac{A''}{A} = \frac{2k_1 T}{\left(k_1 + k_2\right)T - i\omega^2 M} = Se^{i\phi}$$

$$R = \left[\frac{\left(k_1 - k_2\right)^2 T^2 + \omega^4 M^2}{\left(k_1 + k_2\right)^2 T^2 + \omega^4 M^2}\right], \quad \theta = \tan^{-1}\left(\frac{\omega^2 M / T}{k_1 - k_2}\right) + \tan^{-1}\left(\frac{\omega^2 M / T}{k_1 + k_2}\right)$$

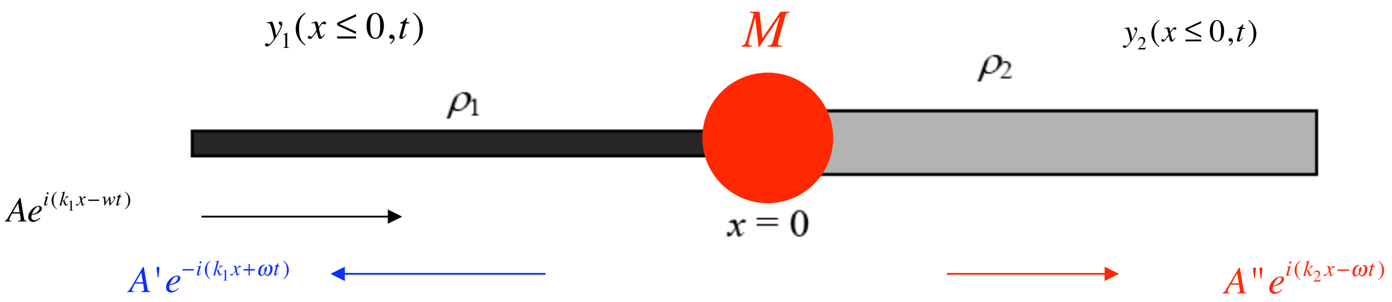
$$S = \left[\frac{4k_1^2 T^2}{\left(k_1 + k_2\right)^2 T^2 + \omega^4 M^2}\right], \quad \phi = \tan^{-1}\left(\frac{\omega^2 M / T}{k_1 + k_2}\right)$$

$$P = \frac{1}{2} T\omega k A^2$$

Power flux

$$|r|^2 + \frac{k_2}{k_1}|t|^2 = R^2 + \frac{k_2}{k_1}S^2 = 1$$

Conservation of energy



$$y_1(x,t) = Ae^{i(k_1 x - wt)} + ARe^{-i(k_1 x + \omega t - \theta)}$$

$$y_2(x,t) = ASe^{i(k_2 x - \omega t + \phi)}$$

$$\frac{A'}{A} = \mathrm{Re}^{i\theta}, \quad \frac{A''}{A} = Se^{i\phi}$$

Real solutions (take A real for simplicity)

$$\mathrm{Re}(y_1) = A\cos(k_1 x - \omega t) + AR\cos(k_1 x + \omega t - \theta)$$

$$\mathrm{Re}(y_2) = AS\cos(k_2 x - \omega t + \phi)$$

Phase shifts

$$|r|^2 + \frac{k_2}{k_1}|t|^2 = R^2 + \frac{k_2}{k_1}S^2 = 1$$

Conservation of energy

REMARKS

- mass M at $x=0 \;\Rightarrow\;$ boundary condition $\partial y_2/\partial x - \partial y_1/\partial x \neq 0$:

$$\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x} = \frac{M}{T}\,\frac{\partial^2 y_1}{\partial t^2} = \frac{M}{T}\,\frac{\partial^2 y_2}{\partial t^2} \qquad \text{at} \;\; x=0$$

- mass M at the boundary produces phase shifts ϕ_r, ϕ_t of the reflected and transmitted waves relative to the incident wave:

$$r = |r| e^{i\phi_r} \quad ; \quad t = |t| e^{i\phi_t}$$

- in the special case $\rho_2 = 0$ (i.e., terminating mass M)

$k_2 = 0$, and $r \to 1$ for $M \to 0$

$r \to -1$ for $M \to \infty$

Impedance

Electrical impedance – a measure of opposition to a time varying electric current

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

$$\left(\tilde{V} = \tilde{I} Z = \tilde{I} R + L \frac{\partial \tilde{I}}{\partial t} + \frac{1}{C} \int \tilde{I} \, dt, \quad \tilde{I} \propto e^{i\omega t} \right)$$

Mechanical impedance – a measure of opposition to motion of a structure subject to a force

$$F(\omega) = Z(\omega) \ \mathrm{v}(\omega)$$

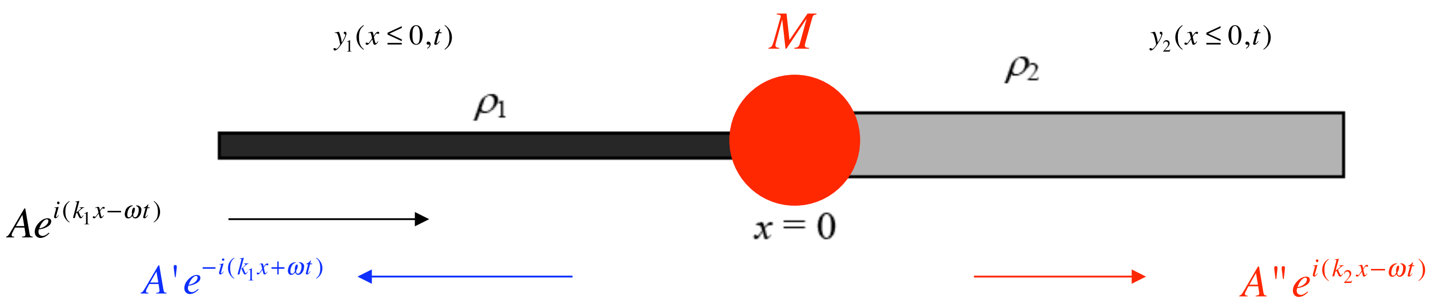
$$Z = \frac{F_y}{\mathrm{v}_y} = \frac{-T \frac{\partial y}{\partial x}}{\frac{\partial y}{\partial t}}$$

$$y(x,t) = A \sin(kx \mp \omega t) \quad \Rightarrow \quad Z_{\pm} = \pm \frac{Tk}{\omega} = \pm \frac{T}{\mathrm{v}}$$

Electromagnetic impedance

Acoustic impedance

Impedance



$$-T\frac{\partial y_1}{\partial x}|_{x=0} + T\frac{\partial y_2}{\partial x}|_{x=0} = M\frac{\partial^2 y_1}{\partial t^2}|_{x=0} = M\frac{\partial^2 y_2}{\partial t^2}|_{x=0}$$

Newton's 2nd law

$$Z_{1\mp}\frac{\partial y_1}{\partial t}|_{x=0} - Z_{2-}\frac{\partial y_2}{\partial t}|_{x=0} = M\frac{\partial^2 y_1}{\partial t^2}|_{x=0} = M\frac{\partial^2 y_2}{\partial t^2}|_{x=0}$$

$$-T\frac{\partial y}{\partial x} = \frac{\partial y}{\partial t}Z_{\mp}$$

$$Z_{1\mp}\mathrm{v}_1 - Z_{2-}\mathrm{v}_2 = M\frac{\partial \mathrm{v}_{i=1,2}}{\partial t} = Z_m \mathrm{v}_1 = Z_m \mathrm{v}_2$$

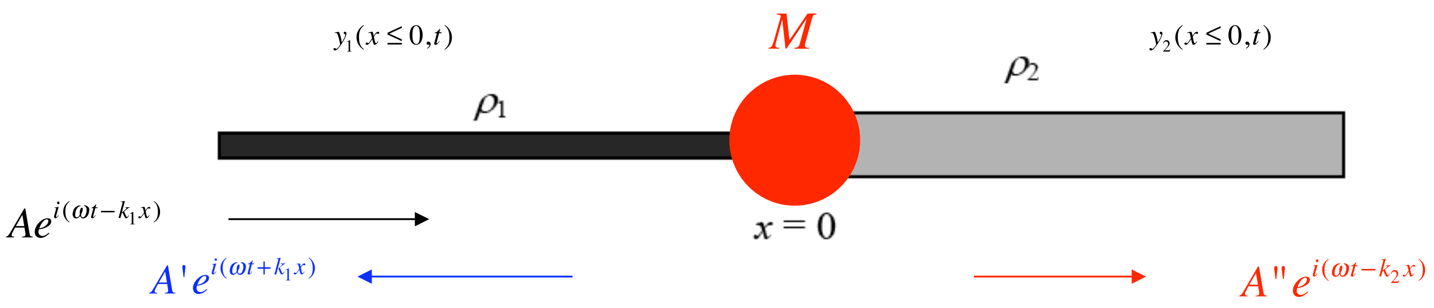
$$Z = \frac{T}{c}, \quad \mathrm{v} = \frac{\partial y}{\partial t}$$

Impedance

$$Z_m = -i\omega M$$

Impedance of mass

Impedance



$$-T\frac{\partial y_1}{\partial x}\Big|_{x=0} + T\frac{\partial y_2}{\partial x}\Big|_{x=0} = M\frac{\partial^2 y_1}{\partial t^2}\Big|_{x=0} = M\frac{\partial^2 y_2}{\partial t^2}\Big|_{x=0}$$

Newton's 2nd law

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$$-T\frac{\partial y}{\partial x} = \frac{\partial y}{\partial t} Z_{\mp}$$

$$Z_{1\mp}\mathrm{v}_1 - Z_{2-}\mathrm{v}_2 = M\frac{\partial \mathrm{v}_{i=1,2}}{\partial t} = Z_m \mathrm{v}_1 \quad = Z_m \mathrm{v}_2$$

$$r = \frac{Z_1 - (Z_2 + Z_m)}{Z_1 + Z_2 + Z_m}$$

$$\boxed{Z_{1-}(A - A') - Z_{2-}A'' = Z_m\left(A + A'\right) = Z_m A''}$$

$$t = \frac{2Z_1}{Z_1 + Z_2 + Z_m}$$

♢ The amount of reflection and transmission that occurs when a traveling wave encounters a boundary is specified entirely in terms of the characteristic impedances presented to the wave by the boundary.

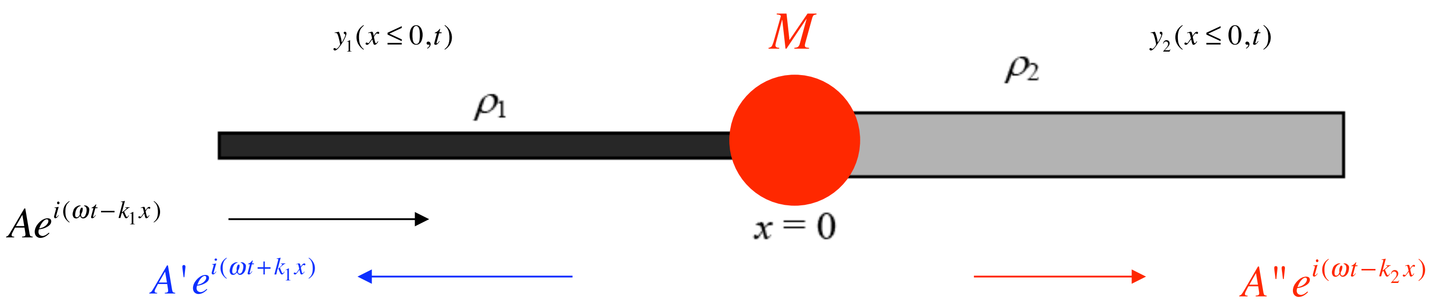
- purely "resistive" impedances

$$Z_1 = \frac{Tk_1}{\omega} \quad ; \quad Z_2 = \frac{Tk_2}{\omega}$$

- impedance of mass

$$Z_m = i\omega M$$

Impedance



$$-T\frac{\partial y_1}{\partial x}\Big|_{x=0} + T\frac{\partial y_2}{\partial x}\Big|_{x=0} = M\frac{\partial^2 y_1}{\partial t^2}\Big|_{x=0} = M\frac{\partial^2 y_2}{\partial t^2}\Big|_{x=0}$$

Newton's 2nd law

$$Z_{1\mp}\frac{\partial y_1}{\partial t}\Big|_{x=0} - Z_{2-}\frac{\partial y_2}{\partial t}\Big|_{x=0} = M\frac{\partial^2 y_1}{\partial t^2}\Big|_{x=0} = M\frac{\partial^2 y_2}{\partial t^2}\Big|_{x=0}$$

$$-T\frac{\partial y}{\partial x} = \frac{\partial y}{\partial t} Z_{\mp}$$

$$Z_{1\mp}\mathrm{v}_1 - Z_{2-}\mathrm{v}_2 = M\frac{\partial \mathrm{v}_{i=1,2}}{\partial t} = Z_m \mathrm{v}_1 = Z_m \mathrm{v}_2$$

$$r = \frac{Z_1 - (Z_2 + Z_m)}{Z_1 + Z_2 + Z_m} = \frac{(k_1 - k_2)T + i\omega^2 M}{(k_1 + k_2)T - i\omega^2 M} = \mathrm{R}e^{i\theta}$$

$$Z_{1-}(A - A') - Z_{2-}A'' = Z_m(A + A') = Z_m A''$$

$$t = \frac{2Z_1}{Z_1 + Z_2 + Z_m} = \frac{2k_1 T}{(k_1 + k_2)T - i\omega^2 M} = Se^{i\phi}$$

Summary

- Wave motion across a boundary or discontinuity can be analyzed in terms of reflected and transmitted waves.

- Nature of discontinuity $\Rightarrow$ specific form of boundary conditions

  Examples:

  — discontinuity in linear mass density

  — discontinuity due to pointlike mass

- Transport of energy across boundary describable by reflection and transmission coefficients

- Useful parameterisation of boundary effects via characteristic impedances

$$Z = \frac{\text{driving force}}{\text{velocity of displacement}}$$