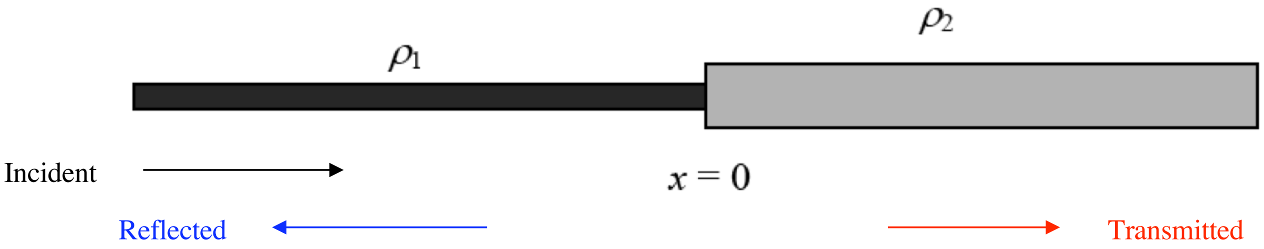
Boundary effects

- What happens to a traveling wave when it encounters a barrier, or discontinuity, or obstacle of some kind

- Example: inhomogeneity in mass density

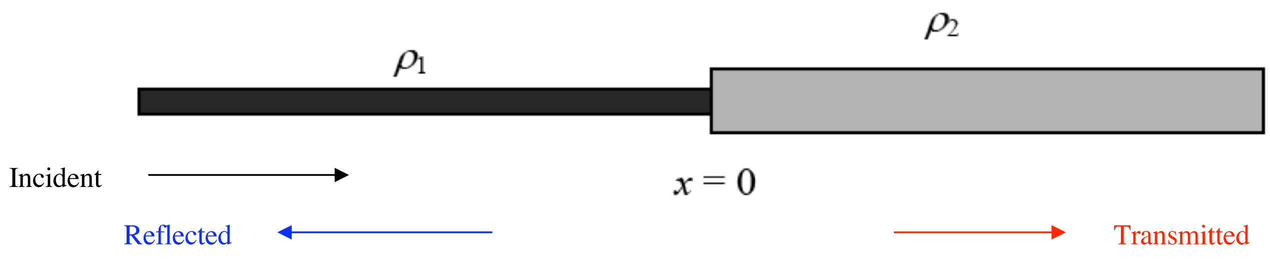


$$\psi_{incident} = A\sin(\omega t - k_1 x) \qquad \mathrm{Im}\{Ae^{i\{k_1 x - wt\}}\} \;\; \text{(A, A', A'' real here)}$$

$$\psi_{reflected} = A'\sin(\omega t + k_1 x) \qquad \mathrm{Im}\{A'e^{-i\{k_1 x + wt\}}\}$$

$$\psi_{transmitted} = A''\sin(\omega t - k_2 x) \qquad \mathrm{Im}\{A''e^{i\{k_2 x - wt\}}\}$$

- All waves have same ω - necessary to satisfy boundary condition at x=0
- Right moving waves $-k_1 x$
- Left moving waves $+k_2 x$



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$$\psi_{transmitted} = A''\sin(\omega t - k_2 x)$$

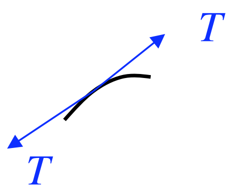
Boundary conditions

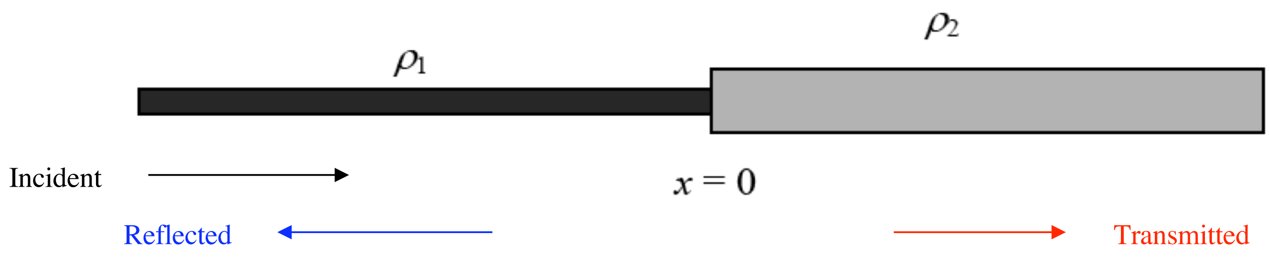
$$y(x = 0-, t) = y(x = 0+, t)$$

String continuous

$$\frac{\partial y}{\partial x}(x = 0-, t) = \frac{\partial y}{\partial x}(x = 0+, t)$$

Forces continuous





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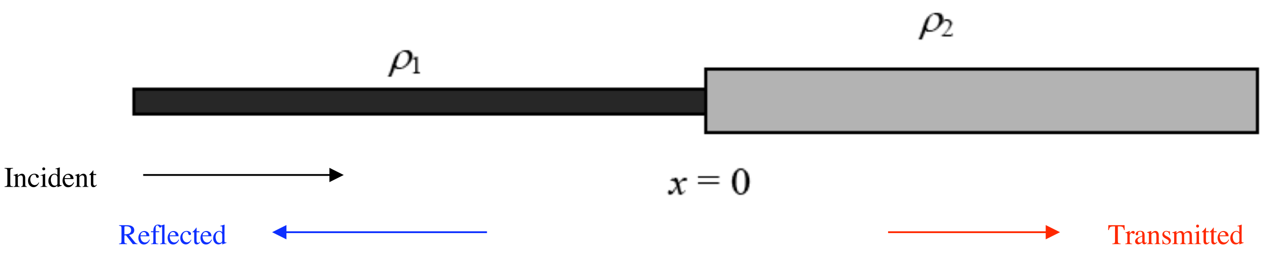
$$\psi_{transmitted} = A''\sin\left(\omega t - k_2 x\right)$$

Boundary conditions

$$y(x=0-,t) = y(x=0+,t) \qquad \psi_{incident}(x=0,t) + \psi_{reflected}(0,t) = \psi_{transmitted}(0,t)$$

$$\frac{\partial y}{\partial x}(x=0-,t) = \frac{\partial y}{\partial x}(x=0+,t) \qquad \frac{\partial \psi_{incident}(x=0,t)}{\partial x} + \frac{\partial \psi_{reflected}(0,t)}{\partial x} = \frac{\partial \psi_{transmitted}(0,t)}{\partial x}$$



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Boundary conditions

$$y(x=0-,t) = y(x=0+,t) \qquad \psi_{incident}(x=0,t) + \psi_{reflected}(0,t) = \psi_{transmitted}(0,t)$$

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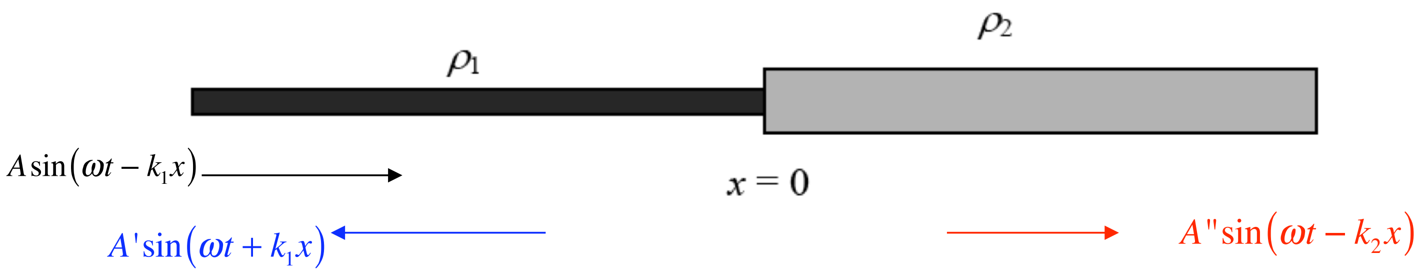
$$A + A' = A''$$

$$k_1\left(A - A'\right) = k_2 A''$$



$$A + A' = A''$$

$$k_1(A - A') = k_2 A''$$

$$r = \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2} \quad \text{reflection amplitude}$$

$$t = \frac{A''}{A} = \frac{2k_1}{k_1 + k_2} \quad \text{transmission amplitude}$$

Special cases:

1) $k_1 = k_2 \Rightarrow \quad A' = 0,\ t = \frac{A''}{A} = 1$ No reflection

2) $k_1 < k_2 \quad \Rightarrow A'$ is negative Reflected wave $= -|A'|\sin(\omega t + k_1 x) = |A'|\sin(\omega t + k_1 x + \pi)$

i.e. PHASE CHANGE at rare-dense boundary $(k_1 < k_2 \Rightarrow \rho_1 < \rho_2)$ $[\mathrm{v} = \omega / k = \sqrt{T/\rho}\,]$

3) $k_1 > k_2 \quad \Rightarrow A'$ is positive

4) $\rho_2 \to \infty \Rightarrow \quad k_2 \to \infty \qquad r = \frac{A'}{A} \to -1 \qquad \mathrm{T} \to 0$ No wave in very heavy string

Energy flux at boundaries

Power flux $P = \frac{1}{2} T\omega k A^2$ (Energy flow per unit time)

Incident power flux $P_I = \frac{1}{2} T\omega k_1 A^2$

Reflected power flux $P_R = \frac{1}{2} T\omega k_1 A'^2$

Transmitted power flux $P_T = \frac{1}{2} T\omega k_2 A''^2$

$$R_P = \frac{P_R}{P_I} = \frac{A'^2}{A^2} = r^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$ Coefficient of reflection

$$T_P = \frac{P_T}{P_I} = \frac{k_2 A''^2}{k_1 A^2} = \frac{k_2}{k_1} t^2 = \frac{k_2}{k_1}\left(\frac{2k_1}{k_1 + k_2}\right)^2 = \frac{4k_1 k_2}{(k_1 + k_2)^2}$$ Coefficient of transmission

$$R_P + T_P = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 + \frac{4k_1 k_2}{(k_1 + k_2)^2} = \frac{k_1^2 + 2k_1 k_2 + k_2^2}{(k_1 + k_2)^2} = 1$$ Conservation of energy

Summary

♠ Continuity of y and $\partial y/\partial x$ at the boundary $x = 0$ determines the amplitudes of the reflected and transmitted waves in terms of the amplitude of the incident wave as a function of the wave numbers k_1 and k_2:

$$r = \frac{k_1 - k_2}{k_1 + k_2} \quad ; \quad t = \frac{2k_1}{k_1 + k_2}$$

♠ Energy transport across the boundary:

$$R \equiv \frac{reflected\ \ flux}{incident\ \ flux} = r^2 \qquad \text{reflection coefficient}$$

$$T \equiv \frac{transmitted\ \ flux}{incident\ \ flux} = \frac{k_2}{k_1}\ t^2 \qquad \text{transmission coefficient}$$

$$1 = R + T$$