Lecture 5 — Waves

◊ Separation of variables method for the wave equation

◊ Application to normal modes of vibrating strings

◊ Example: the plucked string

The wave equation - solution by separation of variables

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 y}{\partial t^2}$$

Try $y(x,t) = X(x)T(t)$

$$T(t)\frac{d^2X(x)}{dx^2} = \frac{1}{\mathrm{v}^2}X(x)\frac{d^2T(t)}{dt^2} \quad \Rightarrow \quad \frac{\ddot{X}}{X} = \frac{1}{\mathrm{v}^2}\frac{\ddot{T}}{T}$$

Hence $\frac{\ddot{X}}{X} = \frac{1}{v^2}\frac{\ddot{T}}{T} = C_S = -k^2$, C_S "Separation Constant" (taken -ve here)

A,B,C,D constants

$$\begin{aligned} \ddot{X} + k^2 X &= 0 \\ \ddot{T} + k^2\mathrm{v}^2 T &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X &= A\sin kx + B\cos kx \\ T &= C\sin k\mathrm{v}t + D\cos k\mathrm{v}t \end{aligned}$$

The wave equation - solution by separation of variables

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k complex possible too

Hence $\frac{\ddot{X}}{X} = \frac{1}{\mathrm{v}^2}\frac{\ddot{T}}{T} = C_S = +k^2$, C_S "Separation Constant" (taken +ve here)

A,B,C,D constants

$$\begin{aligned} \ddot{X} + k^2 X &= 0 \\ \ddot{T} + k^2 v^2 T &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X &= Ae^{kx} + Be^{-kx} \\ T &= Ce^{kvt} + De^{-kvt} \end{aligned}$$

For vibrating string physically relevant case is $-k^2$

$$y(x,t) = X(x)T(t) = \left(A\sin kx + B\cos kx\right)\left(C\sin kvt + D\cos kvt\right)$$
$$= A'\left(\sin kx + B'\cos kx\right)\left(\sin kvt + D'\cos kvt\right)$$

Relation to d'Alembert's solution?

$$y(x,t) = \alpha\cos(kx+\delta)\cos\left(kvt+\rho\right), \qquad (\alpha,\rho,\delta \text{ constants})$$

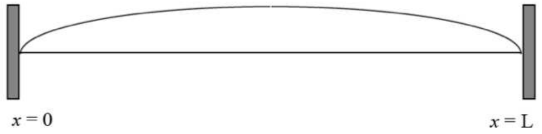
$$= \frac{\alpha}{2}\left[\cos\left(k(x+\mathrm{v}t)+\delta+\rho\right) + \cos\left(k(x-\mathrm{v}t)+\delta-\rho\right)\right]$$

$$\left(= f(x+\mathrm{v}t) + g(x-\mathrm{v}t)\right)$$

Example 1

String with fixed ends initially displaced and at rest

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$



Initial conditions : y(x,t=0) = h(x), $\frac{\partial y}{\partial t}(x, t=0) = 0$

End points: y(0,t)=y(L,t)=0, for all t

$$y(x,t) = (A\sin kx + B\cos kx)(C\sin kvt + D\cos kvt)$$

y(0,t) = 0 $\Rightarrow$ B=0

$$\frac{\partial y}{\partial t}(x, t=0) = 0 \quad \Rightarrow \quad C = 0$$

Eigenvalue equation

$$y(L,t) = 0 \quad \Rightarrow \quad kL = n\pi, \quad n \in Z$$

Principle of superposition

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L}$$

A_n constants

• Normal frequencies:

$$\omega_n = v k_n = \frac{n \pi v}{L} = n \omega_1 \qquad (n \ \ \text{integer})$$

$$\omega_1 = \frac{\pi v}{L} = \frac{\pi}{L} \sqrt{\frac{T}{\rho}} \qquad \text{fundamental frequency}$$

$$\omega_n \ \ , \ \ n > 1 \qquad \text{harmonics}$$

• infinite, discretized set of normal modes

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L}$$

A_n fixed by initial conditions

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = h(x)$$

eg(a) $h(x) = B\sin(m\pi x / L)$ $A_m = B, \quad A_{n\neq m} = 0$

$$y(x,t) = B\sin\frac{m\pi x}{L}\cos\frac{m\pi vt}{L}$$

Standing wave - single normal mode excited

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L}$$

A_n fixed by initial conditions

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = h(x)$$

eg(b) $h(x) = \sin\frac{\pi x}{L} + \frac{1}{2}\sin\frac{2\pi x}{L}$ $\qquad A_1 = 1, \quad A_2 = \frac{1}{2}, \quad A_{n\neq 1,2} = 0$

$$y(x,t) = \sin\frac{\pi x}{L}\cos\frac{\pi vt}{L} + \frac{1}{2}\sin\frac{2\pi x}{L}\cos\frac{2\pi vt}{L}$$

N.B. For non-NM initial displacement, subsequent motion is NOT EQUAL to initial displacement $\times$ varying amplitude.

Shorter wavelengths oscillate faster (constant speed) -Hence shape of wave varies during oscillation.

$$y(x,t)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L}$$

“Fourier series” (2nd year)

A_n fixed by initial conditions

$$y(x,0)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = h(x)$$

General solution

$$A_n = \frac{2}{L}\int_0^L h(x)\sin\frac{n\pi x}{L}\,dx$$

Follows from orthogonality of sines

$$\frac{2}{L}\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, \text{ if } m \neq n \\ 1, \text{ if m=n} \end{cases}$$

ENERGIES OF NORMAL MODES

$$y_n(x,t) = A \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v t}{L} + \delta\right)$$

$$\text{kinetic}: \; K_n = \frac{1}{2}\,\rho \int_0^L dx\, \left(\frac{\partial y_n}{\partial t}\right)^2$$

$$= \frac{1}{2}\rho A^2 \left(\frac{n\pi v}{L}\right)^2 \sin^2\left(\frac{n\pi v t}{L} + \delta\right)\int_0^L dx\, \sin^2\left(\frac{n\pi x}{L}\right)$$

$$= \frac{\rho A^2 n^2 \pi^2 v^2}{4L}\sin^2\left(\frac{n\pi v t}{L} + \delta\right)$$

$$\text{potential}: \; U_n = \frac{1}{2}\,T \int_0^L dx\, \left(\frac{\partial y_n}{\partial x}\right)^2$$

$$= \frac{1}{2} T A^2 \left(\frac{n\pi}{L}\right)^2 \cos^2\left(\frac{n\pi v t}{L} + \delta\right)\int_0^L dx\, \cos^2\left(\frac{n\pi x}{L}\right)$$

$$= \frac{T A^2 n^2 \pi^2}{4L}\cos^2\left(\frac{n\pi v t}{L} + \delta\right)$$

$$\rho v^2 = T \;\Rightarrow E_n = K_n + U_n = \frac{\rho A^2 n^2 \pi^2 v^2}{4L} = \frac{\rho L A^2 \omega_n^2}{4}$$

- What is the energy of the system in terms of the energies of the normal modes?

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi v t}{L} + \delta\right)$$

▷ When evaluating K and U for this $\sum$, all crossed terms give zero because

$$\int_0^L dx\ \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = 0 \qquad \text{if}\ \ n \neq m$$

$$\text{Therefore} \quad K = \sum_n K_n \quad , \qquad U = \sum_n U_n$$

$$\Rightarrow\ E = \sum_n E_n \qquad \textit{sum of the energies of the normal modes}$$