Lecture 4

Waves

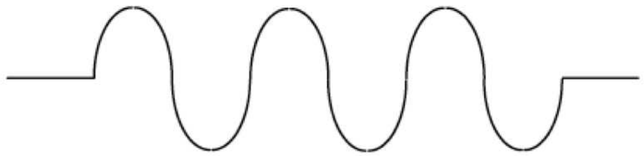
$\diamondsuit$ Phase velocity and group velocity. Dispersion.

$\diamondsuit$ Energy and transport of energy by a wave

Information transmission

Plane wave does not convey information - must be modulated

e.g.



$$y = A\sin\left(kx - \omega t\right), \qquad \left|kx - \omega t\right| \le \omega T / 2$$

$$= 0, \qquad \left|kx - \omega t\right| > \omega T / 2$$

To construct this modulated wave need a wave packet made up of an infinite number of frequencies - Fourier series (2nd year topic)

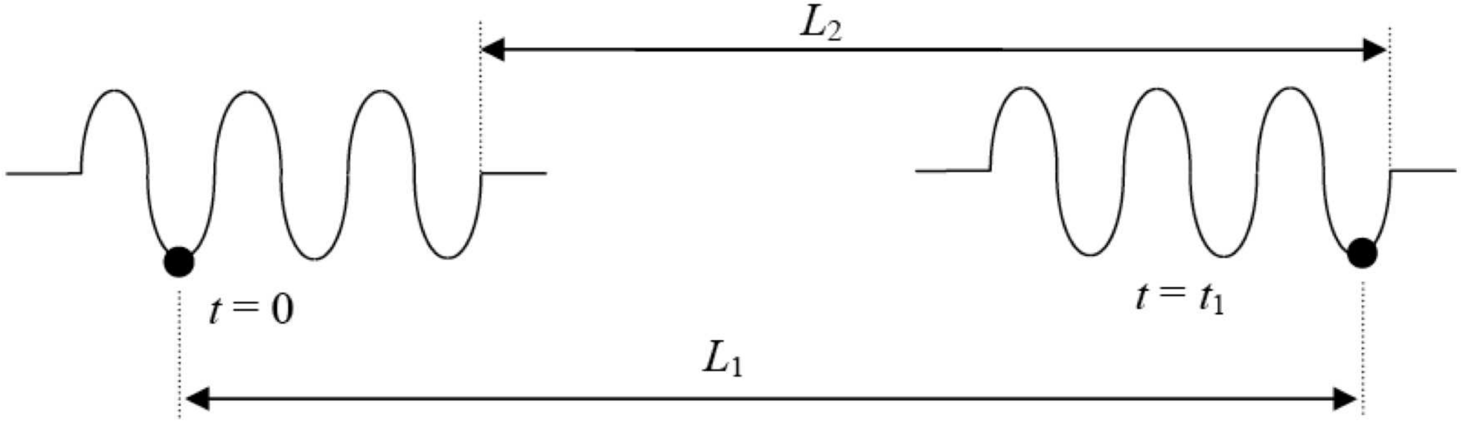
$$y(x,t) = \sum_{n=1}^{N} D_n \cos\left(k_n x - \omega_n t\right)$$

Rate if information transfer : velocity of envelope … "GROUP VELOCITY"

c.f. speed of individual waves of definite frequency … "PHASE VELOCITY" $\left(\mathrm{v} = \frac{\omega_n}{k_n}\right)$

Group and phase velocity

$$y(x,t) = \sum_{n=1}^{N} D_n \cos\left(k_n x - \omega_n t\right)$$



Group velocity, g....... $g = \frac{L_2}{t_1}$

Phase velocity, v..... *e.g.* $v = \frac{L_1}{t_1}$ (Since many frequencies need not be the same as the group velocity)

Dispersion: variation of wave speed with frequency (or wave number)

$$y(x,t) = \sum_{n=1}^{N} D_n \cos\left(k_n x - \omega_n t\right)$$

Non-dispersive medium

$$\mathrm{v_n} = \mathrm{v}$$

$$y(x,t) = \sum_{n=1}^{N} D_n \cos k_n \left(x - \mathrm{v}t\right) \equiv y(x - \mathrm{v}t) \quad \Rightarrow \quad \omega_n = k_n \mathrm{v} \quad \Rightarrow \quad \frac{\delta\omega}{\delta k} \equiv \frac{\omega}{k} = \mathrm{v}$$

$$\boxed{i.e.\ g = \mathrm{v}}$$

e.g Light through glass prism
Different colours ➔ different angles, because the refractive index μ (= c/v) depends on colour (ω), i.e.v depends on ω.

Dispersive medium

$$\mathrm{v_n} \neq \mathrm{v}$$

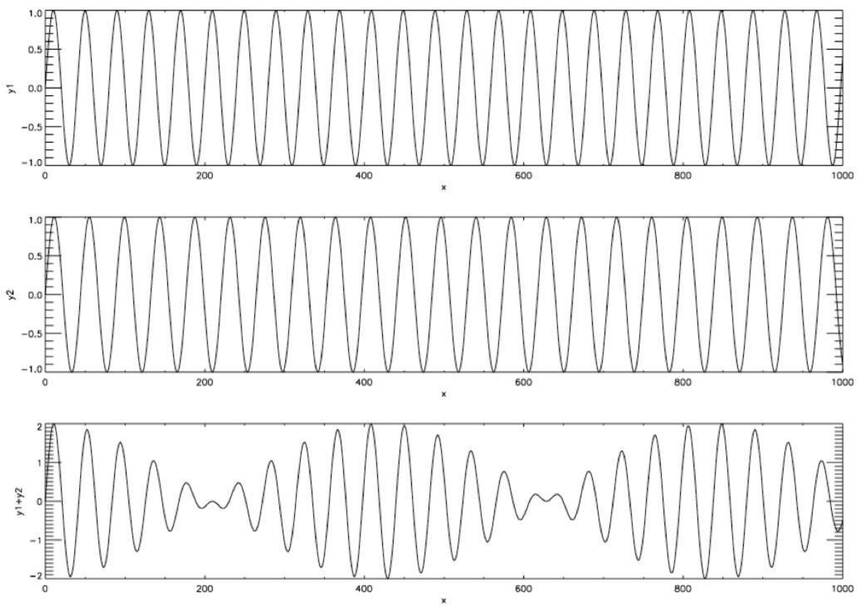
$$\boxed{i.e.\ g \neq \mathrm{v}}$$

- Simple example : superposition of two waves

$$y_1 = A\sin\left[\left(k+\delta k\right)x - \left(\omega+\delta\omega\right)t\right]$$

$$y_2 = A\sin\left[\left(k-\delta k\right)x - \left(\omega-\delta\omega\right)t\right]$$

$$\delta k << k, \quad \delta\omega << \omega$$



$$y = y_1 + y_2 = 2A\cos\left(\delta k.x - \delta\omega.t\right)\sin\left(kx-\omega t\right)$$

Phase velocity $\mathrm{v} \simeq \dfrac{\omega}{\mathrm{k}}$ Group velocity $g = \dfrac{\delta\omega}{\delta k}$ (velocity of envelope)

- Superposition of travelling waves $y_1(x,t)$ and $y_2(x,t)$:

$$y_1 = A\sin[(k+\delta k)x - (\omega+\delta\omega)t] \ , \quad y_2 = A\sin[(k-\delta k)x - (\omega-\delta\omega)t]$$

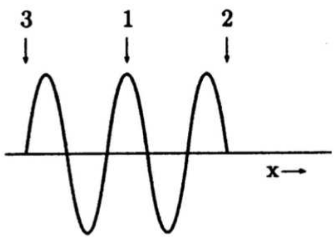
- Using $\sin\alpha + \sin\beta = 2\sin[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$, the resultant wave is

$$y = y_1 + y_2 = 2A\sin(kx-\omega t)\cos(\delta k\ x - \delta\omega\ t)$$

♠ 1st factor $\sin$ varies with frequency ω and wave number k,
i.e., close to the original waves y_1 and y_2,
and corresponding speed $v = \omega/k$ (<u>phase velocity</u>).
♠ 2nd factor $\cos$ varies much more slowly,
with frequency $\delta\omega$ and wave number δk
$\Rightarrow$ amplitude modulation, moving at speed $v_g = \delta\omega/\delta k$ (<u>group velocity</u>).
The modulating envelope encloses a group of short waves.

$$\text{For} \ \ \delta\omega\,,\ \delta k \to 0\,, \ \ v_g = \frac{d\omega}{dk}$$

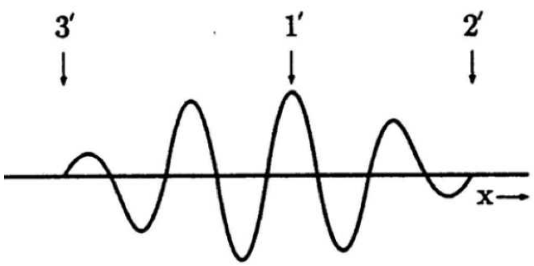
To construct a finite pulse need a superposition of an infinite number of waves of different frequencies



If distribution is peaked around ω, k

$$\text{Phase velocity } \mathrm{v} \simeq \frac{\omega}{\mathrm{k}} \qquad \text{Group velocity } g = \frac{d\omega}{dk} \text{ (velocity of envelope)}$$

In a non-dispersive medium pulse maintains its shape $\frac{d\omega}{dk} = \frac{\omega}{k} = \text{constant}$



In a dispersive medium pulse spreads out
… but mean position moves with group velocity

$N.B. \quad g \leq c$

There are many equivalent expressions for the group velocity:

$$g = \frac{d\omega}{dk}$$

$$\omega = \mathrm{v}k \qquad\qquad g = \mathrm{v} + k\frac{d\mathrm{v}}{dk}$$

$$k = 2\pi / \lambda \qquad\qquad g = \mathrm{v} - \lambda\frac{d\mathrm{v}}{d\lambda}$$

$$\mathrm{v} = c / \mu \qquad\qquad g = \frac{c}{\mu}\left(1 + \frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right)$$

In terms of wavelength λ' in vacuum

$$g = \mathrm{v}\left(1 - \frac{1}{1 + \frac{\mathrm{v}}{\lambda'}\frac{d\lambda'}{dv}}\right) \qquad \lambda' = \lambda\frac{c}{\mathrm{v}} \qquad f = \frac{c}{\lambda'} = \frac{\mathrm{v}}{\lambda}$$

distinction between g and v arises
because of dispersion, i.e., dv/dk nonzero

- Waves travelling through a dispersive medium:

$$\text{group velocity} \;\; v_g = \frac{d\omega}{dk} = \frac{d(vk)}{dk} = v + k\frac{dv}{dk}$$

$$= v + \frac{2\pi}{\lambda}\frac{dv}{d\lambda}\left(-\frac{\lambda^2}{2\pi}\right) = v - \lambda\frac{dv}{d\lambda}$$

EXAMPLE

- Suppose the dispersion relation $v = v(\lambda)$ is given by

$$v^2 = c^2 + \lambda^2\omega_0^2$$

$$\text{Then} \;\; 2v\; dv = 2\lambda\; d\lambda\; \omega_0^2 \;\; , \qquad i.e. \quad \frac{dv}{d\lambda} = \frac{\lambda\omega_0^2}{v}$$

$$\text{So} \;\; v_g = v - \lambda\frac{dv}{d\lambda} = v - \frac{\lambda^2\omega_0^2}{v} \;\; , \qquad i.e. \quad v_g = v - \frac{v^2 - c^2}{v} = \frac{c^2}{v} \quad \Longrightarrow \quad v_g v = c^2$$

product of phase and group velocities equals c^2

Example: waves in deep water

$$v \propto \sqrt{\lambda} \quad , \quad \text{i.e.} \quad v = \frac{C}{\sqrt{k}} \quad \text{with} \quad C \quad \text{constant}$$

Since $v = \omega/k$, then $\omega/k = C/\sqrt{k}$

$$\Rightarrow \quad \omega = C\,\sqrt{k}$$

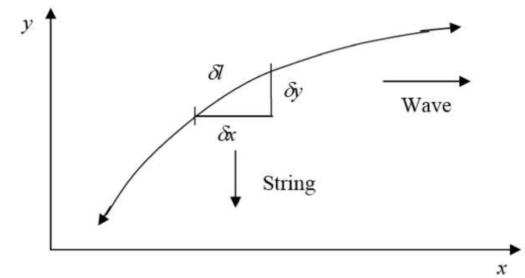
$$\text{Therefore} \quad g = \frac{d\omega}{dk} = \frac{C}{2\sqrt{k}}$$

$$= \frac{1}{2}\,v$$

- group velocity is half the phase velocity:

component wave crests run rapidly through the group, first growing in amplitude and then disappearing

Energy of vibrating string



$$y = A\sin\left(kx - \omega t\right)$$

Kinetic Energy

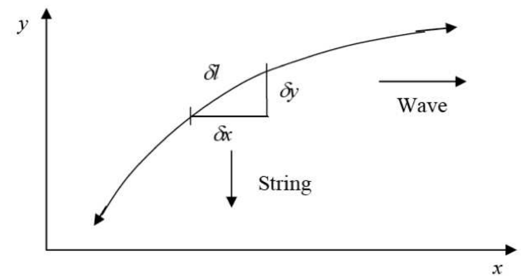
$$KE \text{ of section} = \frac{1}{2}\rho\delta x\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\rho A^2\omega^2\cos^2\left(kx - \omega t\right)\delta x$$

$$KE = \frac{1}{2}\rho A^2\omega^2\int_x^{x+l\lambda}\cos^2\left(kx' - \omega t\right)dx' = \frac{1}{2}\rho A^2\omega^2 \times \frac{l\lambda}{2}$$

(integer l wavelengths)

Energy of vibrating string

$$y = A\sin\left(kx - \omega t\right)$$



$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \equiv \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Kinetic Energy

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$$\boxed{KE / l\lambda = \frac{1}{4}\rho A^2\omega^2}$$

Potential Energy

$$PE \text{ in stretched string element} = T\left(\delta l - \delta x\right) = T\delta x\left(\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2} - 1\right) \approx \frac{1}{2}TA^2k^2\cos^2\left(kx - \omega t\right)\delta x$$

$$PE = \frac{1}{2}TA^2k^2\int_x^{x+l\lambda}\cos^2\left(kx' - \omega t\right)dx'$$

$$\boxed{PE / l\lambda = \frac{1}{4k}TA^2k^2}$$

$\mathrm{v} = \omega / k = \sqrt{T/\rho} \Rightarrow Tk^2 = \rho\omega^2 \Rightarrow$ PE=KE! (Example of virial theorem)

Note

kinetic energy per unit length : $\dfrac{dK}{dx} = \dfrac{1}{2}\,\rho\,\left(\dfrac{\partial y}{\partial t}\right)^2$

potential energy per unit length : $\dfrac{dU}{dx} = \dfrac{1}{2}\,T\,\left(\dfrac{\partial y}{\partial x}\right)^2$

$$\frac{dE}{dx} = \frac{dK}{dx} + \frac{dU}{dx}$$

- For $y(x,t) = f(x-ct)$, $c = \sqrt{T/\rho}$, one has

$$\frac{\partial y}{\partial t} = -cf' \quad , \qquad \frac{\partial y}{\partial x} = f'$$

So $\dfrac{1}{2}\,\rho\,\left(\dfrac{\partial y}{\partial t}\right)^2 = \dfrac{1}{2}\,\rho\,c^2\,(f')^2 = \dfrac{1}{2}\,T\,(f')^2 = \dfrac{1}{2}\,T\,\left(\dfrac{\partial y}{\partial x}\right)^2$

$$\Rightarrow\ \frac{dK}{dx} = \frac{dU}{dx}$$

Energy flow

Total energy per wavelength, E/λ= KE+PE=$\frac{1}{2}\rho A^2\omega^2$

Distance travelled = vt

$$\text{Energy flow/unit time} = \left(\frac{1}{2}\rho A^2\omega^2\right)\mathrm{v}t/t$$

$$= \frac{1}{2}\rho\omega^2 A^2 \mathrm{v}$$

$$= \frac{1}{2}\rho\omega^3 A^2 / k \qquad \mathrm{v} = \frac{\omega}{k}$$

$$= \frac{1}{2}Tk^2 A^2 \mathrm{v} \qquad Tk^2 = \rho\omega^2$$

$$= \frac{1}{2}T\omega k A^2$$