Lecture 4

Waves

- Phase velocity and group velocity. Dispersion.

- Energy and transport of energy by a wave
Information transmission

Plane wave does not convey information - must be modulated

e.g.

\[ y = A \sin(kx - \omega t), \quad |kx - \omega t| \leq \frac{\omega T}{2} \]

\[ = 0, \quad |kx - \omega t| > \frac{\omega T}{2} \]

To construct this modulated wave need a wave packet made up of an infinite number of frequencies - Fourier series (2nd year topic)

\[ y(x,t) = \sum_{n=1}^{N} D_n \cos(k_n x - \omega_n t) \]

Rate if information transfer: velocity of envelope … “GROUP VELOCITY”

c.f. speed of individual waves of definite frequency … “PHASE VELOCITY”  \[ v = \frac{\omega_n}{k_n} \]
Group and phase velocity

\[ y(x,t) = \sum_{n=1}^{N} D_n \cos(k_n x - \omega_n t) \]

Group velocity, \( g \) ....... \( g = \frac{L_2}{t_1} \)

Phase velocity, \( v \) ..... \( e.g. \ v = \frac{L_1}{t_1} \) (Since many frequencies need not be the same as the group velocity)
Dispersion: variation of wave speed with frequency  
(or wave number)

\[ y(x,t) = \sum_{n=1}^{N} D_n \cos(k_n x - \omega_n t) \]

Non-dispersive medium

\[ v_n = v \]

\[ y(x,t) = \sum_{n=1}^{N} D_n \cos k_n (x - vt) \equiv y(x - vt) \quad \Rightarrow \quad \omega_n = k_n v \quad \Rightarrow \quad \frac{\delta \omega}{\delta k} \equiv \frac{\omega}{k} = v \]

i.e. \( g = v \)

Dispersive medium

\[ v_n \neq v \]

i.e. \( g \neq v \)

\[ \text{e.g. Light through glass prism} \]
\[ \text{Different colours } \Rightarrow \text{ different angles, because the} \]
\[ \text{refractive index } \mu (= c/v) \text{ depends on colour } (\omega), \]
\[ \text{i.e. } v \text{ depends on } \omega. \]
Simple example: superposition of two waves

\[ y_1 = A \sin \left[ (k + \delta k) x - (\omega + \delta \omega) t \right] \]

\[ y_2 = A \sin \left[ (k - \delta k) x - (\omega - \delta \omega) t \right] \]

\[ y = y_1 + y_2 = 2A \cos (\delta k x - \delta \omega t) \sin (k x - \omega t) \]

Phase velocity \( v = \frac{\omega}{k} \)

Group velocity \( g = \frac{\delta \omega}{\delta k} \) (velocity of envelope)
- Superposition of travelling waves $y_1(x,t)$ and $y_2(x,t)$:

$$y_1 = A \sin[(k + \delta k)x - (\omega + \delta \omega)t] , \quad y_2 = A \sin[(k - \delta k)x - (\omega - \delta \omega)t]$$

- Using $\sin \alpha + \sin \beta = 2 \sin[(\alpha + \beta)/2] \cos[(\alpha - \beta)/2]$, the resultant wave is

$$y = y_1 + y_2 = 2A \sin(kx - \omega t) \cos(\delta k x - \delta \omega t)$$

♠ 1st factor $\sin$ varies with frequency $\omega$ and wave number $k$, i.e., close to the original waves $y_1$ and $y_2$, and corresponding speed $v = \omega/k$ (phase velocity).

♠ 2nd factor $\cos$ varies much more slowly, with frequency $\delta \omega$ and wave number $\delta k$

$\Rightarrow$ amplitude modulation, moving at speed $v_g = \delta \omega/\delta k$ (group velocity).

The modulating envelope encloses a group of short waves.

For $\delta \omega, \delta k \rightarrow 0$, $v_g = \frac{d\omega}{dk}$
To construct a finite pulse need a superposition of an infinite number of waves of different frequencies.

If distribution is peaked around $\omega, k$

\[
\text{Phase velocity } v = \frac{\omega}{k} \quad \text{Group velocity } g = \frac{d\omega}{dk} \quad \text{(velocity of envelope)}
\]

In a non-dispersive medium pulse maintains its shape

\[
\frac{d\omega}{dk} = \frac{\omega}{k} = \text{constant}
\]

In a dispersive medium pulse spreads out

… but mean position moves with group velocity

\[N.B. \quad g \leq c\]
There are many equivalent expressions for the group velocity:

\[ g = \frac{d\omega}{dk} \]

\[ \omega = vk \]

\[ k = \frac{2\pi}{\lambda} \]

\[ v = \frac{c}{\mu} \]

In terms of wavelength \( \lambda' \) in vacuum

\[ g = v \left( 1 - \frac{1}{1 + \frac{\lambda}{\lambda'} \frac{d\mu}{d\lambda}} \right) \]

\[ \lambda' = \lambda \frac{c}{v} \quad f = \frac{c}{\lambda'} = \frac{v}{\lambda} \]

*distinction between g and v arises because of dispersion, i.e., \( dv/dk \) nonzero*
Waves travelling through a dispersive medium:

\[ v_g = \frac{d\omega}{dk} = \frac{d(vk)}{dk} = v + k \frac{dv}{dk} \]

\[ = v + 2\pi \frac{dv}{\lambda} \left( -\frac{\lambda^2}{2\pi} \right) = v - \lambda \frac{dv}{d\lambda} \]

**EXAMPLE**

Suppose the dispersion relation \( v = v(\lambda) \) is given by

\[ v^2 = c^2 + \lambda^2 \omega_0^2 \]

Then \( 2v \ dv = 2\lambda \ d\lambda \ \omega_0^2 \), i.e. \( \frac{dv}{d\lambda} = \frac{\lambda \omega_0^2}{v} \)

So \( v_g = v - \lambda \frac{dv}{d\lambda} = v - \frac{\lambda^2 \omega_0^2}{v} \), i.e. \( v_g = v - \frac{v^2 - c^2}{v} = \frac{c^2}{v} \) \( \implies v_g v = c^2 \)

product of phase and group velocities equals \( c^2 \)
Example: waves in deep water

\[ v \propto \sqrt{\lambda} , \quad \text{i.e.} \quad v = \frac{C}{\sqrt{k}} \quad \text{with} \quad C \quad \text{constant} \]

Since \( v = \omega / k \), then \( \omega / k = C / \sqrt{k} \)

\[ \Rightarrow \quad \omega = C \sqrt{k} \]

Therefore \[ g = \frac{d\omega}{dk} = \frac{C}{2\sqrt{k}} \]

\[ = \frac{1}{2} v \]

- group velocity is half the phase velocity: component wave crests run rapidly through the group, first growing in amplitude and then disappearing
Energy of vibrating string

\[ y = A \sin(kx - \omega t) \]

Kinetic Energy

\[ KE \text{ of section} = \frac{1}{2} \rho \delta x \left( \frac{\partial y}{\partial t} \right)^2 = \frac{1}{2} \rho A^2 \omega^2 \cos^2(kx - \omega t) \delta x \]

\[ KE = \frac{1}{2} \rho A^2 \omega^2 \int_x^{x + l\lambda} \cos^2(kx' - \omega t)dx' = \frac{1}{2} \rho A^2 \omega^2 \times \frac{l\lambda}{2} \quad \text{(integer l wavelengths)} \]
Energy of vibrating string

\[ y = A \sin(kx - \omega t) \]

Kinetic Energy

\[ KE \text{ of section} = \frac{1}{2} \rho \delta x \left( \frac{\partial y}{\partial t} \right)^2 = \frac{1}{2} \rho A^2 \omega^2 \cos^2(kx - \omega t) \delta x \]

\[ KE = \frac{1}{2} \rho A^2 \omega^2 \int_x^{x+l} \cos^2(kx' - \omega t) dx' = \frac{1}{2} \rho A^2 \omega^2 \times \frac{l}{2} \]

Potential Energy

\[ PE \text{ in stretched string element} = T \left( \delta l - \delta x \right) = T \delta x \left( \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} - 1 \right) \approx \frac{1}{2} TA^2 k^2 \cos^2(kx - \omega t) \delta x \]

\[ PE = \frac{1}{2} TA^2 k^2 \int_x^{x+l} \cos^2(kx' - \omega t) dx' \]

\[ v = \omega / k = \sqrt{T / \rho} \Rightarrow Tk^2 = \rho \omega^2 \Rightarrow PE = KE! \] (Example of virial theorem)
kinetic energy per unit length: \[ \frac{dK}{dx} = \frac{1}{2} \rho \left( \frac{\partial y}{\partial t} \right)^2 \]

potential energy per unit length: \[ \frac{dU}{dx} = \frac{1}{2} T \left( \frac{\partial y}{\partial x} \right)^2 \]

\[ \frac{dE}{dx} = \frac{dK}{dx} + \frac{dU}{dx} \]

- For \( y(x, t) = f(x - ct) \), \( c = \sqrt{T/\rho} \), one has

\[ \frac{\partial y}{\partial t} = -cf' , \quad \frac{\partial y}{\partial x} = f' \]

So \[ \frac{1}{2} \rho \left( \frac{\partial y}{\partial t} \right)^2 = \frac{1}{2} \rho c^2 (f')^2 = \frac{1}{2} T (f')^2 = \frac{1}{2} T \left( \frac{\partial y}{\partial x} \right)^2 \]

\[ \Rightarrow \frac{dK}{dx} = \frac{dU}{dx} \]
Energy flow

Total energy per wavelength, $E/\lambda = KE + PE = \frac{1}{2} \rho A^2 \omega^2$

Distance travelled = $vt$

Energy flow/unit time = \[\left(\frac{1}{2} \rho A^2 \omega^2\right)v t / t\]

\[= \frac{1}{2} \rho \omega^2 A^2 v\]

\[= \frac{1}{2} \rho \omega^3 A^2 / k\] \[v = \frac{\omega}{k}\]

\[= \frac{1}{2} Tk^2 A^2 v\] \[Tk^2 = \rho \omega^2\]

\[= \frac{1}{2} T \omega k A^2\]