The Wave Equation

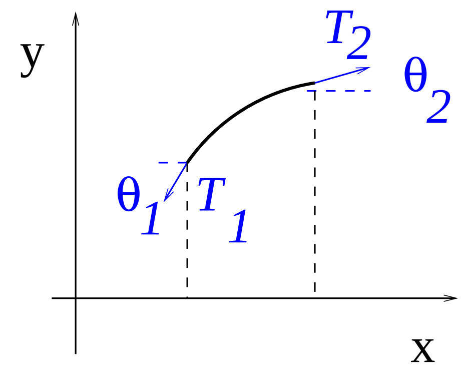
♢ The method of characteristics

♢ Inclusion of boundary conditions

♢ Traveling waves and stationary waves

TRANSVERSE OSCILLATIONS OF AN ELASTIC STRING

- uniform linear density ρ
- small displacements y



- $T_1 \cos\theta_1 = T_2 \cos\theta_2$

  for small θ, $\cos\theta \simeq 1 \;\Rightarrow\; T_1 = T_2 = T$

- $\rho\ \delta x\ \dfrac{\partial^2 y}{\partial t^2} = T\sin\theta_2 - T\sin\theta_1$

$$\sin\theta \simeq \tan\theta \simeq \frac{\partial y}{\partial x} \quad\Longrightarrow\quad \rho\ \delta x\ \frac{\partial^2 y}{\partial t^2} = T \underbrace{\left[(\frac{\partial y}{\partial x})_2 - (\frac{\partial y}{\partial x})_1 \right]}_{(\partial^2 y/\partial x^2)\delta x + \ldots}$$

Thus $\quad \rho\ \dfrac{\partial^2 y}{\partial t^2} = T\ \dfrac{\partial^2 y}{\partial x^2}$

i.e. , $\quad \dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2} \quad , \quad c^2 \equiv \dfrac{T}{\rho} \qquad$ wave equation

2nd-order linear PDEs

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = R(x,y)$$

- $B^2 - AC < 0$ elliptic. Ex.: Laplace eqn. $u_{xx} + u_{yy} = 0$
- $B^2 - AC = 0$ parabolic. Ex.: diffusion eqn. $u_t - \alpha u_{xx} = 0$
- $B^2 - AC > 0$ hyperbolic. Ex.: wave eqn. $u_{tt} - c^2 u_{xx} = 0$

$\diamondsuit$ General solutions of PDEs depend on arbitrary functions

(analogous to solutions of ODEs depending on arbitrary constants)

$\longrightarrow$ boundary conditions to determine such functions

- Cauchy boundary conditions:

assign function u and normal derivative $\partial u/\partial n$ on given curve γ in xy plane

(relevant to hyperbolic PDEs)

D'Alembert's solution

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Change variables

$$u = x - ct$$
$$v = x + ct$$

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = -c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \Rightarrow \quad \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 y = \frac{1}{c^2}\left(-c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}\right)^2 y = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)^2 y$$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial u \partial v} = 0}$$

f, g arbitrary functions

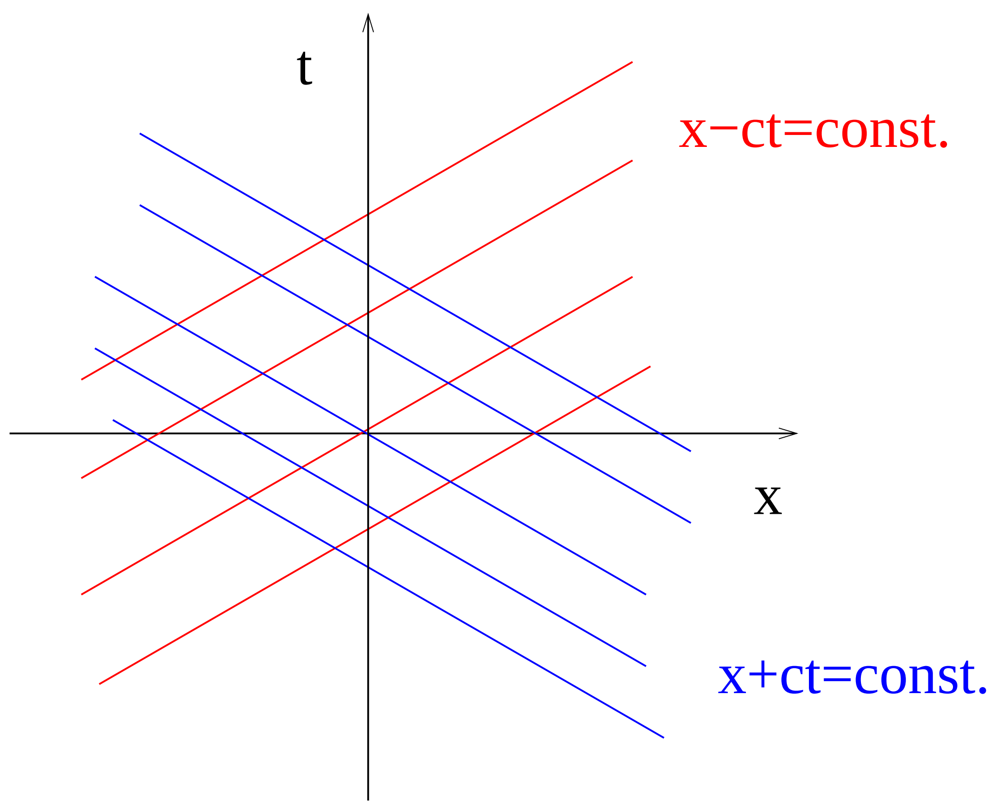
General solution $y(u,v) = f(u) + g(v)$ *i.e.* $y(x,t) = f(x - ct) + g(x + ct)$

• The curves in the xt plane

$$x - ct = \text{const.}$$

$$x + ct = \text{const.}$$

are called the *characteristics* of the wave equation.



CHARACTERISTICS

Consider the PDE $\quad Ay_{tt} + 2By_{tx} + Cy_{xx} + Dy_t + Ey_x + Fy = R$

$$Q = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \quad \text{matrix of 2nd} - \text{order coefficients}$$

- Characteristics of the above PDE are defined as the curves

$$\chi(t, x) = \text{const.}$$

such that their normal n is rotated by 90° by Q or is annihilated by Q, i.e.,

$$n \cdot Q \; n = 0.$$

- Since $n \propto \nabla\chi$, for characteristics $\quad \nabla\chi \cdot Q \; \nabla\chi = 0.$

▷ Example. Wave eqn. : $-\dfrac{1}{c^2} y_{tt} + y_{xx} = 0 \;\Rightarrow\; Q = \begin{pmatrix} -1/c^2 & 0 \\ 0 & 1 \end{pmatrix}$

The curves $\chi_{\mp}(t, x) = x \mp ct =$const. are the characteristics of the wave equation because

$$\nabla\chi_{\mp} = \begin{pmatrix} \mp c \\ 1 \end{pmatrix} \;\Rightarrow\; \nabla\chi \cdot Q \; \nabla\chi = (\, \mp c \quad 1 \,) \begin{pmatrix} -1/c^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mp c \\ 1 \end{pmatrix} = (\, \mp c \quad 1 \,) \begin{pmatrix} \pm 1/c \\ 1 \end{pmatrix} = 0$$

• The condition $\nabla\chi \cdot Q\ \nabla\chi = 0$ implies, using $\nabla\chi = (\chi_t, \chi_x)$, that

$$\begin{pmatrix} \chi_t & \chi_x \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \chi_t \\ \chi_x \end{pmatrix} = 0 ,$$

$$\text{i.e.,} \quad A\chi_t^2 + 2B\chi_t\chi_x + C\chi_x^2 = 0 .$$

Expressing the derivatives in terms of $x'(t) = -\chi_t/\chi_x$,

$$A[x'(t)]^2 - 2Bx'(t) + C = 0 .$$

♠ Hyperbolic eqns. $(B^2 - AC > 0)$ have 2 families of characteristics

♠ Parabolic eqns. $(B^2 - AC = 0)$ have 1 $(Q\ \nabla\chi = 0)$

♠ Elliptic eqns. $(B^2 - AC < 0)$ have none

Uses of characteristics

- Characteristics $\chi_{\mp}(t,x) =$ const. can be used to solve hyperbolic equations by means of the transformation of variables

$$u = \chi_{-}(t,x)$$

$$v = \chi_{+}(t,x)$$

$\triangleright$ Example: D'Alembert solution of the wave equation

- Characteristics serve to analyze whether boundary value problems for PDEs are well posed.

$\triangleright$ Example: Cauchy conditions on curve γ well-defined provided γ is not a characteristic

[Cauchy-Kovalevska theorem]

Cauchy boundary conditions and characteristics

Consider the PDE $\quad A y_{tt} + 2B y_{tx} + C y_{xx} = H(y_t, y_x, y, t, x)$

- Cauchy conditions: Suppose y and the normal derivative y_n are assigned on the curve γ specified by

$$G(t, x) = 0$$

▷ The normal and tangential directions to γ are $n \propto \nabla G = (G_t, G_x)$, $\tau \propto (-G_x, G_t)$.

▷ Given y on γ, we can compute tangential derivative y_τ. From y_n and y_τ we can get y_t and y_x.

- Can we determine higher derivatives as well?

$$\frac{\partial}{\partial \tau} y_t = \tau \cdot \nabla\ y_t \propto \begin{pmatrix} -G_x & G_t \end{pmatrix} \begin{pmatrix} y_{tt} \\ y_{xt} \end{pmatrix} = -G_x y_{tt} + G_t y_{xt}$$

$$\frac{\partial}{\partial \tau} y_x = \tau \cdot \nabla\ y_x \propto \begin{pmatrix} -G_x & G_t \end{pmatrix} \begin{pmatrix} y_{tx} \\ y_{xx} \end{pmatrix} = -G_x y_{tx} + G_t y_{xx}$$

$\Rightarrow$ 3 linear equations in y_{tt}, y_{tx}, y_{xx}, with unique solution if det $\neq 0$

$$\det \begin{pmatrix} A & 2B & C \\ -G_x & G_t & 0 \\ 0 & -G_x & G_t \end{pmatrix} \neq 0$$

$$\Rightarrow \; AG_t^2 + 2BG_tG_x + CG_x^2 \neq 0$$

That is, Cauchy conditions on curve γ are well-defined provided γ is not a characteristic

D'Alembert's solution

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad \Rightarrow \quad y(x,t) = f(x-ct) + g(x+ct)$$

f and g are determined by the initial conditions:

Suppose at time $t = 0$, the wave has an initial displacement $U(x)$ and an initial velocity $V(x)$

$$y(x,0) = f(x) + g(x) = U(x)$$

$$\frac{\partial y(x,0)}{\partial t} = -cf'(x) + cg'(x) = V(x) \quad \Rightarrow \quad f(x) - g(x) = -\frac{1}{c}\int_b^x V(x')\,dx'$$

$$f(x) = \frac{1}{2}U(x) - \frac{1}{2c}\int_b^x V(x')\,dx'$$

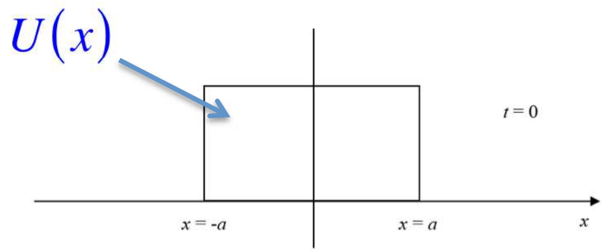
$$g(x) = \frac{1}{2}U(x) + \frac{1}{2c}\int_b^x V(x')\,dx'$$

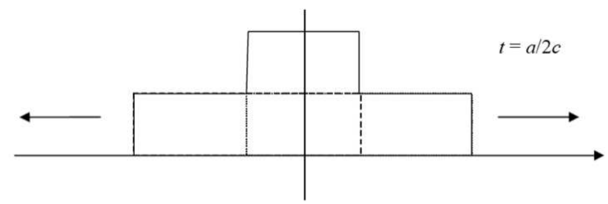
$$\boxed{y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\left[\int_b^{x+ct} V(x)\,dx - \int_b^{x-ct} V(x)\,dx\right] = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(x)\,dx}$$

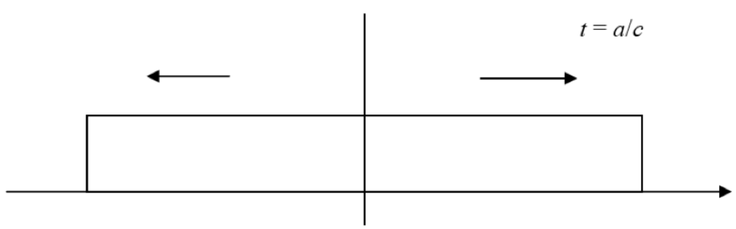
Ex. Wave with initial rectangular displacement released from rest, $V(x)=0$

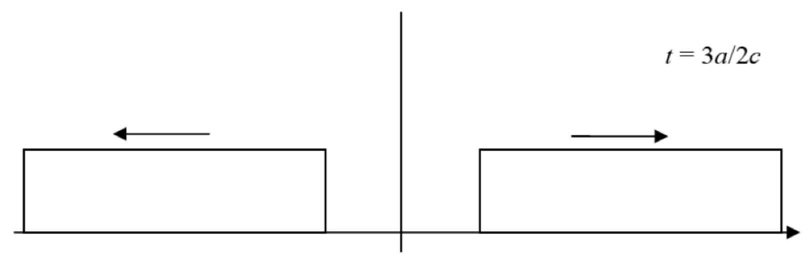
$$y(x,t)=\frac{1}{2}\left[U(x-ct)+U(x+ct)\right]$$

$$\left(y(x,t)=\frac{1}{2}\left[U(x-ct)+U(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}V(x)\,dx\right)$$

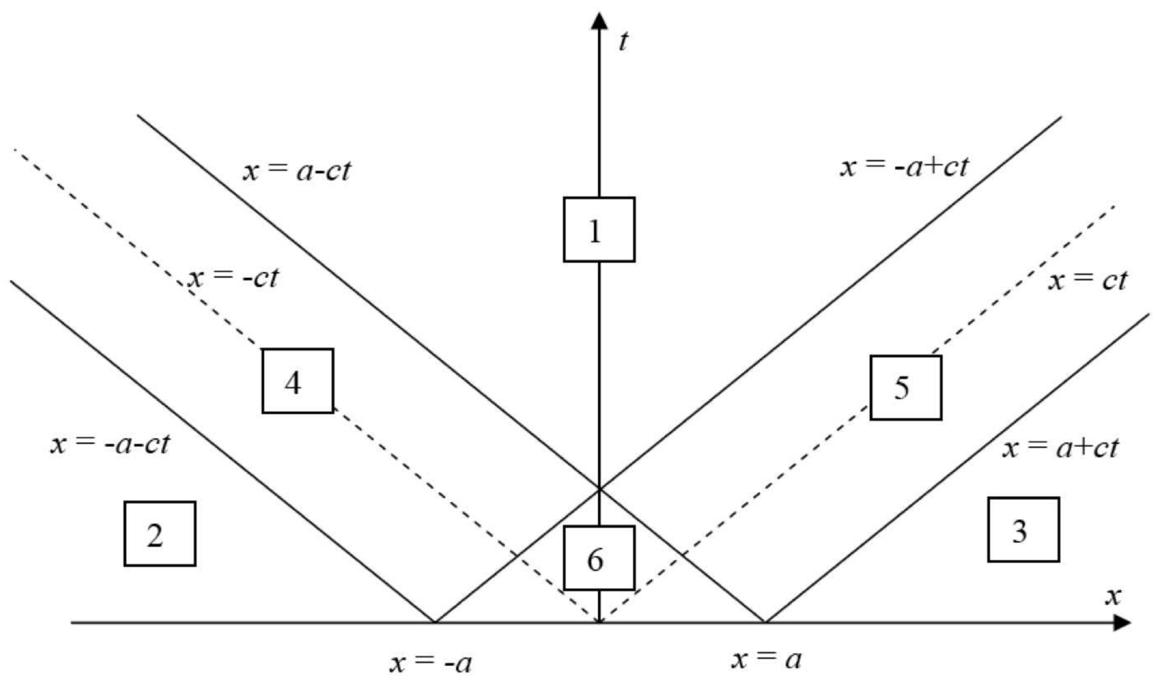








This figure can also be represented on an (x,t) domain. Let $y(x,t)$ be pointing out of the paper.



In regions 1, 2, and 3, $y(x,t) = 0$ for all x, t

In region 5, $y(x,t) = \frac{1}{2}u(x-ct) \qquad -a \le x-ct \le a$

In region 4, $y(x,t) = \frac{1}{2}u(x+ct) \qquad -a \le x+ct \le a$

In region 6, $y(x,t) = \frac{1}{2}\left[u(x-ct)+u(x+ct)\right] \qquad x-ct > -a$

$$x+ct < a$$

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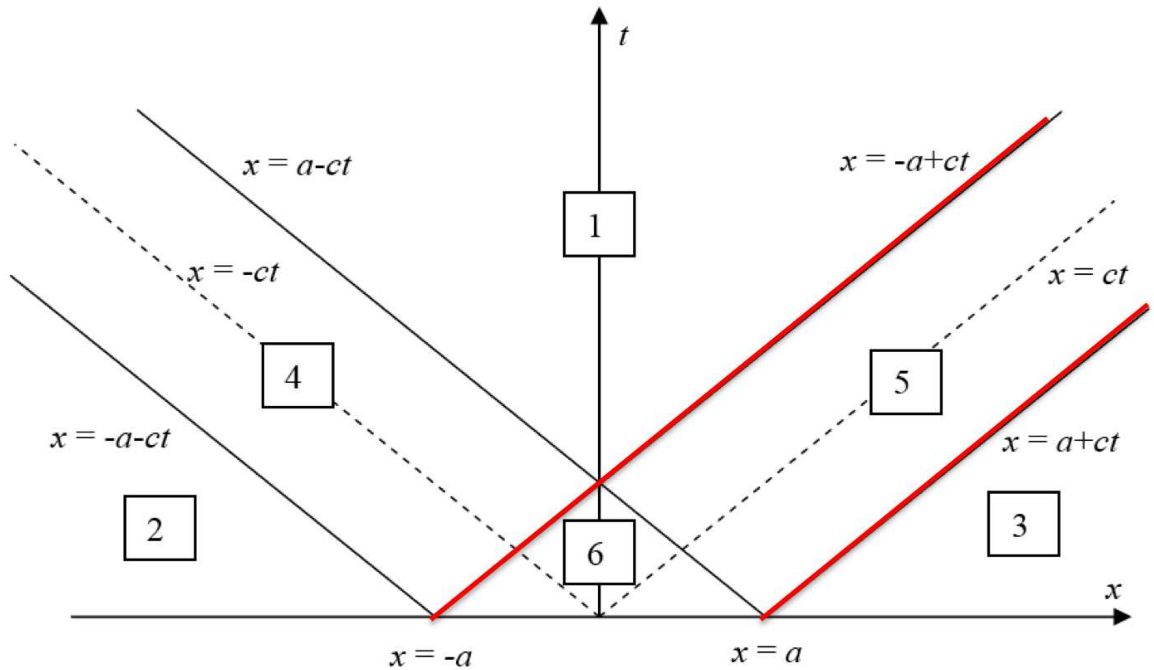

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In region 6, $y(x,t) = \frac{1}{2}\left[u(x-ct)+u(x+ct)\right]$ $\quad x - ct > -a$

$x + ct < a$

$$\boxed{y(x,t) = f(x-ct) + g(x+ct)}$$

What is the form of $f(x)$, $g(x)$?

$$\boxed{y(x,t) = f(x - ct) + g(x + ct)}$$

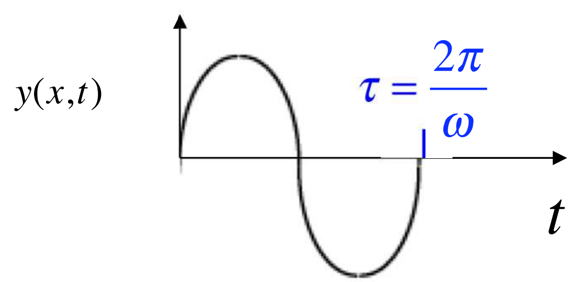
If time dependence is $\cos(\omega t)$ the full (x,t) dependence is given by

$$\boxed{y(x,t) = A\cos(kx + \omega t) + B\cos(kx - \omega t)}$$

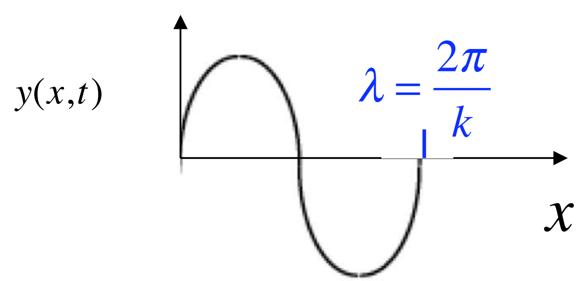
constant k

- Speed of wave $c = \dfrac{\omega}{k}$

- Frequency $f = \dfrac{1}{\tau} = \dfrac{\omega}{2\pi}$



- Wavelength $\lambda = \dfrac{2\pi}{k}$



k is "wavenumber"

We can write the equation of a travelling wave in a number of analogous forms:

	Velocity	Wavelength	Period	Angular frequency
$A\sin(kx-\omega t)$	ω / k	$2\pi / k$	$2\pi / \omega$	ω
$A\sin k(x-vt)$	v	$2\pi / k$	$2\pi / vk$	vk
$A\sin\left[2\pi\left(\frac{x}{\lambda}-\frac{t}{\tau}\right)\right]$	λ / τ	λ	τ	$2\pi / \tau$
$A\sin\left[2\pi(x-vt)/\lambda\right]$	v	λ	λ / v	$2\pi v / \lambda$

N.B. Can include phase most easily by putting

$$y(x,t)=\mathrm{Re}\left[A\exp\left[i(kx-\omega t)\right]\right]$$

where A is complex.

N.B.2 Sometimes more convenient to switch x and t, i.e.

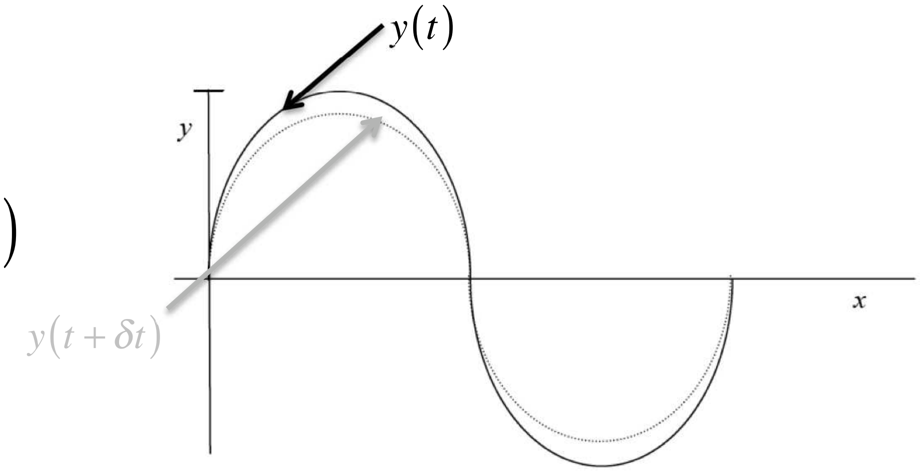
$$y(x,t)=A\sin(\omega t-kx)$$

This is still a travelling wave moving to the right.

For non-sinusoidal wave moving to right with speed v, can always write as $f(x-vt)$.

Stationary waves

$$y = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$
$$= 2A\sin kx\cos\omega t$$



"stationary wave"

More generally

$$y = A\sin(kx - \omega t + 2\delta_1) + A\sin(kx + \omega t + 2\delta_2)$$
$$= 2A\sin(kx + \delta_1 + \delta_2)\cos(\omega t + \delta_1 - \delta_2)$$

Electromagnetic waves

Maxwell's equations (free space)

$$\nabla.\vec{E} = 0$$

$$\nabla.\vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} + \nabla\overbrace{\left(\nabla.\vec{E}\right)}^{0}$$

$$-\nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial \nabla \times \vec{B}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$EM \text{ plane wave } \vec{E} = \vec{E}(z)$$

$$\boxed{\frac{\partial^2 \vec{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}}$$

$$\frac{\partial \vec{E}}{\partial z} = 0 \Rightarrow \quad \boxed{E_z = 0}$$

Transverse wave

Polarisation

$$E_x = A\sin(kx - \omega t)$$
$$E_y = B\sin(kx - \omega t + \phi)$$

A	B	ϕ	Polarisation state	
1	0	-	Linear	→
0	1	-	Linear	↑
1	1	0	Linear	↗
1	1	π	Linear	↖
1	1	$\pi/2$	Circular	(LH)
1	1	$-\pi/2$	Circular	(RH)