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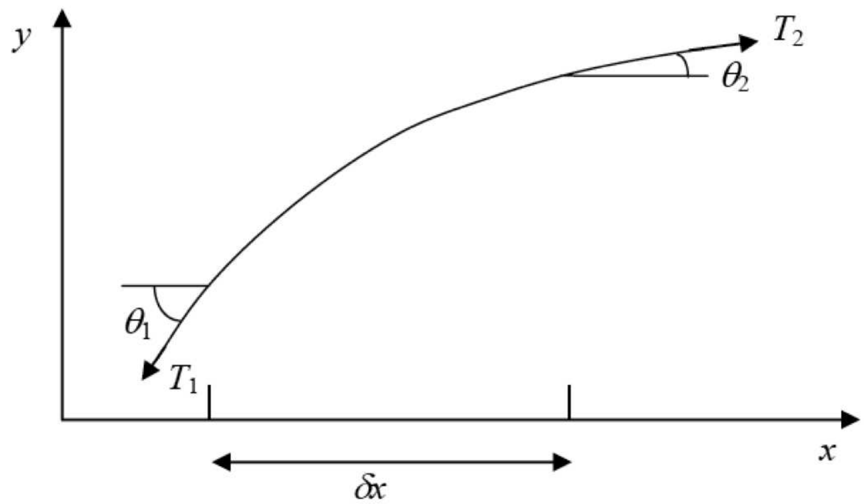
- The Wave Equation

Transverse displacements of an elastic string of linear density (kg/m) ρ



Resolve horizontal forces : $T_1 \cos\theta_1 = T_2 \cos\theta_2$ for small θ, $\cos\theta \sim 1 \Rightarrow T_1 = T_2 = T$

Resolve vertical forces $T \sin\theta_2 - T \sin\theta_1 = \left(\rho \delta x\right) \dfrac{\partial^2 y}{\partial t^2}$

$$\boxed{\therefore T\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \rho \delta x \frac{\partial^2 y}{\partial t^2}}$$

$$\left(\sin\theta \approx \tan\theta = \frac{\partial y}{\partial x}\right)$$

$$\boxed{T\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \rho\delta x \frac{\partial^2 y}{\partial t^2}}$$

$$\left(\frac{\partial y}{\partial x}\right)_2 = \left(\frac{\partial y}{\partial x}\right)_1 + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\delta x + \ldots \quad \Rightarrow \quad T\left(\frac{\partial^2 y}{\partial x^2}\right)\delta x = \rho\frac{\partial^2 y}{\partial t^2}\delta x$$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \equiv \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}}$$

This is a WAVE EQUATION with velocity $c = \sqrt{T/\rho}$
(hence larger tension or lighter string leads to faster waves)

Remarks

- Style of analysis similar to study of N oscillating masses on a string;

  instead of N coordinates $y_p(t)$, $p = 1, \ldots, N$

  now we have $y(x,t)$, $x =$ continuous variable

- Motion given by partial differential equation (PDE) in x and t:

  the wave equation.

  2nd-order linear PDE of hyperbolic type with constant coefficients

- Recall from N oscillators case: onset of wave behavior from looking at sinusoidal dependence on p

2nd-order linear PDEs

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = R(x,y)$$

- $B^2 - AC < 0$ elliptic. Ex.: Laplace eqn. $u_{xx} + u_{yy} = 0$
- $B^2 - AC = 0$ parabolic. Ex.: diffusion eqn. $u_t - \alpha u_{xx} = 0$
- $B^2 - AC > 0$ hyperbolic. Ex.: wave eqn. $u_{tt} - c^2 u_{xx} = 0$

$\diamondsuit$ General solutions of PDEs depend on arbitrary functions

(analogous to solutions of ODEs depending on arbitrary constants)

$\longrightarrow$ boundary conditions to determine such functions

- Cauchy boundary conditions:

assign function u and normal derivative $\partial u / \partial n$ on given curve γ in xy plane

(relevant to hyperbolic PDEs)

D'Alembert's solution

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Change variables

$$u = x - ct$$
$$v = x + ct$$

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = -c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \Rightarrow \quad \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 y = \frac{1}{c^2}\left(-c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}\right)^2 y = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)^2 y$$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial u \partial v} = 0}$$

f, g   arbitrary functions

General solution $\quad y(u,v) = f(u) + g(v) \quad i.e. \quad y(x,t) = f(x - ct) + g(x + ct)$

$\Diamond$ D'Alembert realizes that $y_{xx} = c^{-2} y_{tt}$ implies that
y is a function of $x - ct$ $+$ a function of $x + ct$

$\Diamond$ Interpretation of D'Alembert analysis in terms of traveling waves:

- Take e.g. the part of the solution $y = f(x - ct)$.

At time $t = t_1$: $y(x, t_1) = f(x - ct_1)$

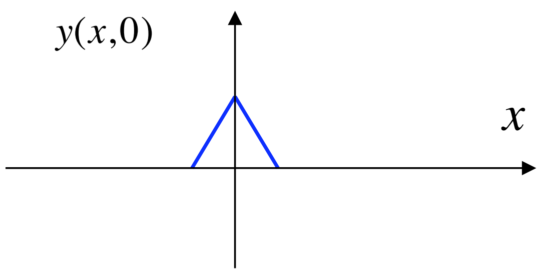
At time $t_2 > t_1$: $y(x, t_2) = f(x - ct_2) = f[x - ct_1 - c(t_2 - t_1)]$

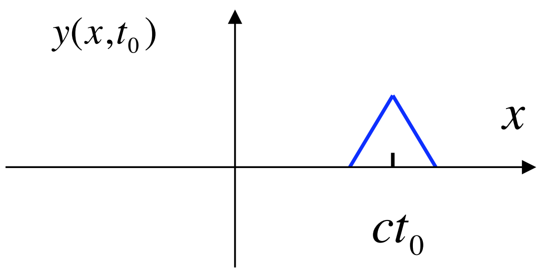
$$= f[\underbrace{(x - c(t_2 - t_1))}_{x'} - ct_1]$$

i.e., y at time t_2 and position x is the same as y was
at time t_1 and position $x' = x - c(t_2 - t_1)$ shifted leftwards
$\Rightarrow$ wave has traveled to the right with speed c

Travelling waves

$$y\left(x,t\right)=f\left(x-ct\right)$$

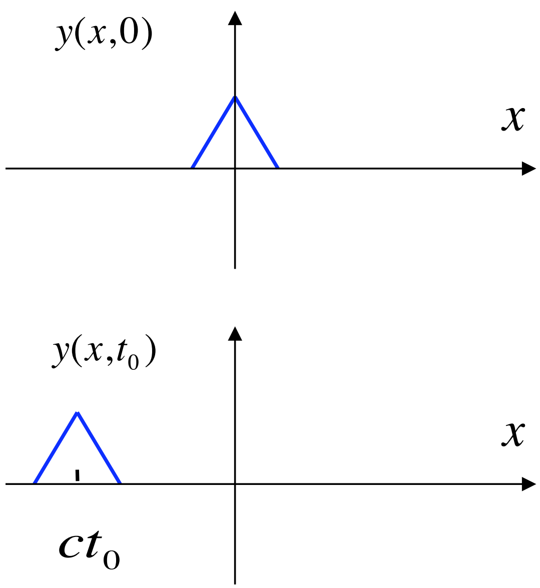




Wave moves to right with speed c

Travelling waves

$$y(x,t) = g(x+ct)$$



Wave moves to left with speed c

Summary

- transverse displacements $y(x,t)$ of elastic string

$\longrightarrow$ wave equation $\quad y_{xx} = c^{-2} y_{tt}$

- hyperbolic 2nd-order linear PDE

- solution of wave equation $\quad y(x,t) = f(x-ct) + g(x+ct)$

traveling waves

- next: determine f and g from initial conditions