# Part $\mathbf{B}$. Waves

◊ Large variety of physical phenomena

▷ unifying mathematical structure: the wave equation

◊ New kind of DE: partial differential equation (PDE)

- We will study *continuous* systems.
- Introductory example: $N$ coupled oscillators (*arbitrarily large* $N$)

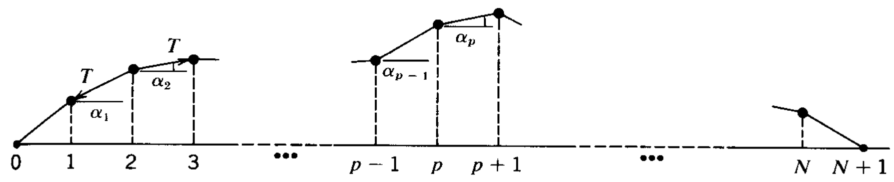
# Waves

Waves are everywhere :

| | |
|---|---|
| Strings | Violin |
| Membranes | Drum |
| Air | Sound |
| Optics | Geometrical optics, interference and diffraction |
| E.M. | Radio, T.V.,…. |
| Quantum Mechanics | Wave/particle, uncertainty principle, ß decay… |
| Seismology | Earthquakes |

# N coupled oscillators

Consider the transverse oscillations of $N$ particles of mass $m$ spaced equally along a flexible, elastic, massless string, which is under tension $T$.




Assume the particles are displaced by small distances $y_i$ :

$\alpha_i$ small

$\sin\alpha_i \simeq \alpha_i \simeq \tan\alpha_i$

$\cos\alpha_i \simeq 1 - \frac{1}{2}\alpha_i^2 \simeq 1$

$$l' = l/\cos\alpha \approx l \quad \Rightarrow \quad \text{tension in string constant, T}$$

$$F_x = -T\cos\alpha_{p-1} + T\cos\alpha_p \simeq 0$$

$$F_y = -T\sin\alpha_{p-1} + T\sin\alpha_p \approx -\frac{T}{l}\left(y_p - y_{p-1}\right) + \frac{T}{l}\left(y_{p+1} - y_p\right) = m\ddot{y}_p$$

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} - y_{p-1}\right) = 0, \qquad \omega_0^2 = T/ml$$

## N coupled oscillators

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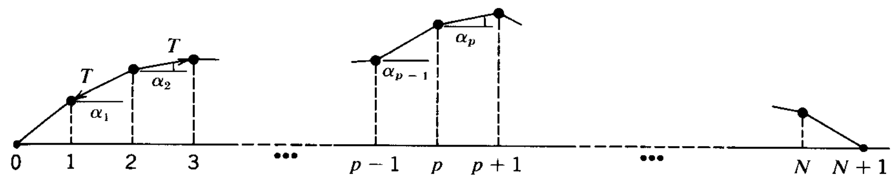



Assume the particles are displaced by small distances $y_i$ :

Consider the $p^{\text{th}}$ particle

$$F_y = -T\sin\alpha_{p-1} + T\sin\alpha_p \approx -\frac{T}{l}\left(y_p - y_{p-1}\right) + \frac{T}{l}\left(y_{p+1} - y_p\right)$$

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} - y_{p-1}\right) = 0, \qquad \omega_0^2 = T/ml$$

General solution :

Superposition of N normal modes

$$y_p = \sum_{n=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right)\left(D_n \cos\left(\omega_n t + \delta_n\right)\right)$$

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right]$$

E.G. N=2

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} - y_{p-1}\right) = 0, \qquad \omega_0^2 = T/ml$$

$$\ddot{y}_1 + 2\omega_0^2 y_1 - \omega_0^2\left(y_2\right) = 0,$$

$$\ddot{y}_2 + 2\omega_0^2 y_2 - \omega_0^2\left(y_1\right) = 0$$

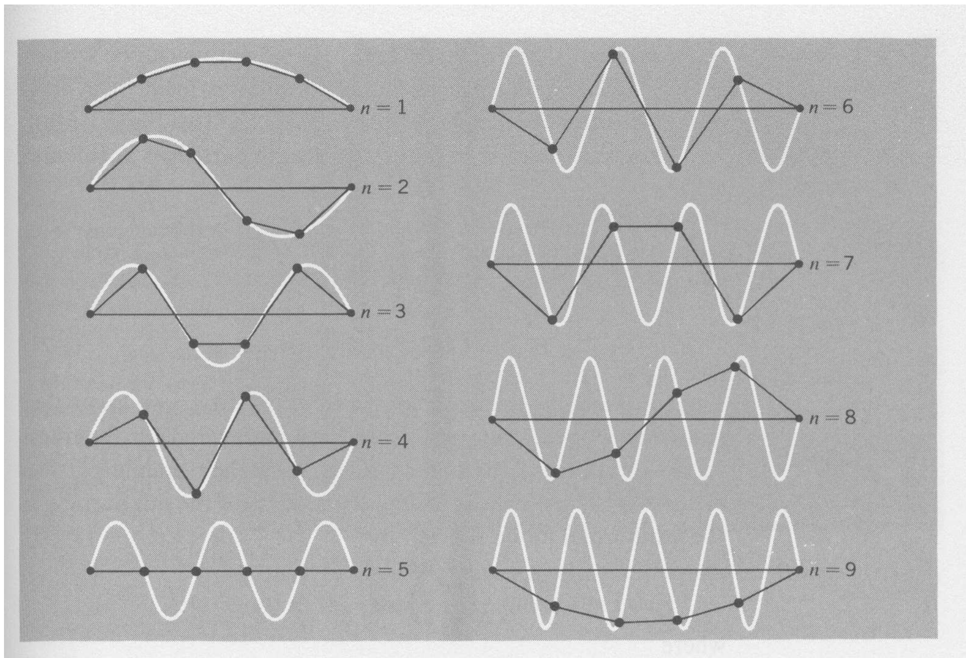
$$\ddot{q}_1 + \omega_0^2 q_1 = 0 \qquad q_1 = y_1 + y_2$$

$$\ddot{q}_2 + 3\omega_0^2 q_2 = 0 \qquad q_2 = y_1 - y_2$$

$$\left(c.f.\ y_p = \sum_{n=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right)\left(D_n \cos\left(\omega_n t + \delta_n\right)\right), \quad \omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2\left(N+1\right)}\right], N = 2\right)$$

E.G. N=4

$$y_p = \sum_{n=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right)\left(D_n \cos\omega_n t + E_n \sin\omega_n t\right), \quad \omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right], \quad N = 4$$




(reproduced from French, 1971)

Clearly there 4 normal modes in all. Note that $n = 6, 7, 8, 9$ repeat patterns of $n = 4, 3, 2, 1$ with opposite sign.

In the limit $N \to \infty$ one obtains the wave equation for transverse waves

# To sum up:

- $N$ normal frequencies of the system of $N$ coupled oscillators $p = 1, \ldots, N$:

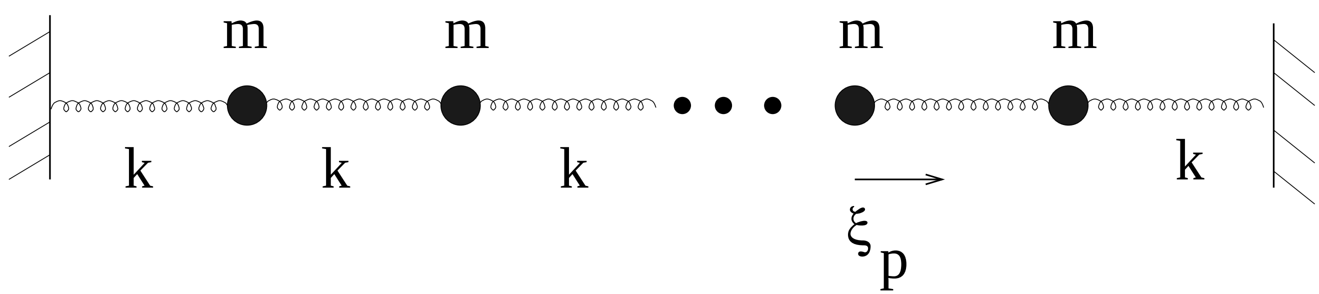
$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \quad , \qquad n = 1, \ldots, N$$

- General solution for $y_p(t)$ is superposition of $N$ normal modes with coefficients

$$Y_{p,n} = c_n \sin\left(\frac{pn\pi}{N+1}\right) \quad , \qquad n = 1, \ldots, N$$

- transverse displacements of every particle in every mode fall on a sinusoidal curve in $p$ illustrating the start of wave pattern

# System of N masses with longitudinal oscillations




$\xi_p$ longitudinal displacement from equilibrium for particle $p$

$$\ddot{\xi}_p + 2\omega_0^2 \xi_p - \omega_0^2 (\xi_{p+1} + \xi_{p-1}) = 0 \ , \quad p = 1, \ldots, N$$

$$\text{where } \ \omega_0^2 = k/m$$

$\Rightarrow$ same normal modes as in previous example