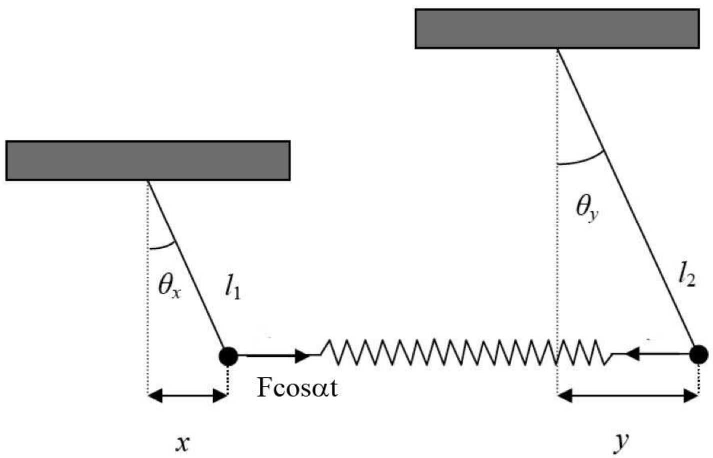
Lecture 4

Normal Modes

$\diamondsuit$ Coupled driven oscillators

$\diamondsuit$ Double pendulum

The damped driven pendulum



$$m_1\ddot{x} = -\gamma\dot{x} - m_1 g x / l_1 + k\left(y - x\right) + F\cos\alpha t$$

$$m_2\ddot{y} = -\gamma\dot{y} - m_2 g y / l_2 - k\left(y - x\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{\gamma}{m_1}\frac{d}{dt} + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2} + \frac{\gamma}{m_2}\frac{d}{dt} + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{Re}\left(e^{i\alpha t}\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2}+\frac{\gamma}{m_1}\frac{d}{dt}+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2}+\frac{\gamma}{m_2}\frac{d}{dt}+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

CF

$$\begin{pmatrix} x \\ y \end{pmatrix}=\mathrm{Re}\left(\begin{pmatrix} X \\ Y \end{pmatrix}e^{i\omega t}\right)$$

$$\begin{pmatrix} -\omega^2+i\frac{\gamma}{m_1}\omega+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2+i\frac{\gamma}{m_2}\omega+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (1)$$

$$\begin{vmatrix} -\omega^2+i\frac{\gamma}{m_1}\omega+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2+i\frac{\gamma}{m_2}\omega+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{vmatrix}=0$$

Eigenvalue eq.

Substitute (complex) eigenvalues in (1) to obtain eigenvectors

Simple case $m_1 = m_2 = m \quad l_1 = l_2 = l$

$$\begin{vmatrix} -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right) \end{vmatrix} = 0$$

$$-\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}\right) = 0 \qquad \text{or} \qquad -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{2k}{m}\right) = 0$$

$$\bar{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2}$$ Eigenvalues

$$\omega_1^2 = \frac{g}{l} \quad \text{or} \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

$\gamma = 0$ eigenvalues (c.f. previous result)

Eigenvectors

$$\begin{pmatrix} -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}\right) = 0 \qquad \text{or} \qquad -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{2k}{m}\right) = 0$$

$$\overline{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2}$$

Eigenvalues

$\omega = \overline{\omega}_1$

$$\begin{pmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = 0$$

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = A_1 e^{i\phi_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\omega = \overline{\omega}_2$

$$\begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = A_2 e^{i\phi_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Eigenvectors

Eigenvectors

$$\begin{pmatrix} -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}\right) = 0 \qquad \text{or} \qquad -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}+\frac{2k}{m}\right) = 0$$

$$\overline{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2}$$

Eigenvalues

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(A_1 e^{i\phi_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{\gamma}{2m}t} e^{it\sqrt{\omega_1^2-\left(\frac{\gamma}{2m}\right)^2}}\right) + \mathrm{Re}\left(A_2 e^{i\phi_2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{\gamma}{2m}t} e^{it\sqrt{\omega_2^2-\left(\frac{\gamma}{2m}\right)^2}}\right)$$

$$= e^{-\frac{\gamma}{2m}t}\left(A_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\left(t\sqrt{\omega_1^2-\left(\frac{\gamma}{2m}\right)^2}+\phi_1\right) + A_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos\left(t\sqrt{\omega_2^2-\left(\frac{\gamma}{2m}\right)^2}+\phi_2\right)\right)$$

Decoupling method

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x) - \gamma\dot{x}$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x) - \gamma\dot{y}$$

$$m(\ddot{x}+\ddot{y}) = -m\frac{g}{l}(x+y) - \gamma(\dot{x}+\dot{y})$$

$$m(\ddot{x}-\ddot{y}) = (-m\frac{g}{l} - 2k)(x-y) - \gamma(\dot{x}-\dot{y})$$

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \omega_1^2 q_1 = 0$$

$$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \omega_2^2 q_2 = 0$$

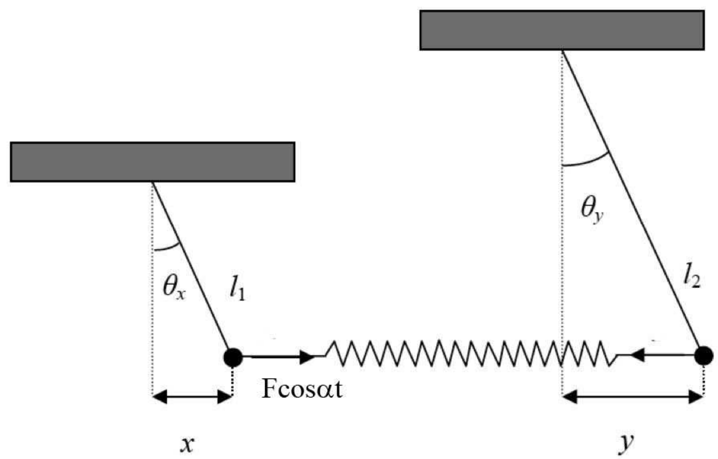
Two 2nd order differential equations - techniques discussed last term give

$$q_1 = A_1\cos(\omega_1't + \phi_1)e^{-\alpha t} \qquad \omega_1' = \sqrt{\omega_1^2 - \alpha^2}$$

$$q_2 = A_2\cos(\omega_2't + \phi_2)e^{-\alpha t} \qquad \omega_2' = \sqrt{\omega_2^2 - \alpha^2}$$

$$\alpha = \frac{\gamma}{2m}$$

The damped driven pendulum - the Particular Integral



$$m_1\ddot{x} = -\gamma\dot{x} - m_1 g x / l_1 + k\left(y - x\right) + F\cos\alpha t$$
$$m_2\ddot{y} = -\gamma\dot{y} - m_2 g y / l_2 - k\left(y - x\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{\gamma}{m_1}\frac{d}{dt} + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2} + \frac{\gamma}{m_2}\frac{d}{dt} + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2}+\frac{\gamma}{m_1}\frac{d}{dt}+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2}+\frac{\gamma}{m_2}\frac{d}{dt}+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

PI

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\begin{pmatrix} P \\ Q \end{pmatrix} e^{i\alpha t}\right)$$

$$\begin{pmatrix} -\alpha^2+i\frac{\gamma}{m_1}\alpha+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2+i\frac{\gamma}{m_2}\alpha+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} \equiv \mathbf{MP} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{P} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} = \mathbf{M}^{-1}\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} -\alpha^2+i\frac{\gamma}{m_1}\alpha+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2+i\frac{\gamma}{m_2}\alpha+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\mathbf{M}^{-1}\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\alpha t}\right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\mathbf{M}^{-1} \frac{F}{m_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\alpha t} \right)$$

$$\mathbf{M} = \begin{pmatrix} -\alpha^2 + i\frac{\gamma}{m_1}\alpha + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2 + i\frac{\gamma}{m_2}\alpha + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix}$$

$$e.g. \quad m_1 = m_2 = m \quad l_1 = l_2 = l \quad \gamma = 0$$

$$\left(\omega_1^2 = \frac{g}{l}, \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m} \right)$$

$$\mathbf{M} = \begin{pmatrix} -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) \end{pmatrix}$$

$$\mathbf{M}^{-1} = \frac{1}{DetM} \begin{pmatrix} -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) \end{pmatrix}$$

$$DetM = \left(-\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) \right)^2 - \left(\frac{k}{m}\right)^2 = \left(\alpha^2 - \omega_1^2\right)\left(\alpha^2 - \omega_2^2\right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{F}{m}\cos\alpha t\right) \frac{1}{\left(\alpha^2 - \omega_1^2\right)\left(\alpha^2 - \omega_2^2\right)} \begin{pmatrix} -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) \\ \frac{k}{m} \end{pmatrix} = \left(\frac{F}{m}\cos\alpha t\right) \frac{1}{\left(\alpha^2 - \omega_1^2\right)\left(\alpha^2 - \omega_2^2\right)} \begin{pmatrix} -\alpha^2 + \left(\omega_2^2 + \omega_1^2\right)/2 \\ \left(\omega_2^2 - \omega_1^2\right)/2 \end{pmatrix}$$

$$= \left(-\frac{F}{2m}\cos\alpha t\right) \begin{pmatrix} \frac{1}{\left(\alpha^2 - \omega_1^2\right)} + \frac{1}{\left(\alpha^2 - \omega_2^2\right)} \\ \frac{1}{\left(\alpha^2 - \omega_1^2\right)} - \frac{1}{\left(\alpha^2 - \omega_2^2\right)} \end{pmatrix}$$

Forced oscillations - decoupling method

$$m_1 = m_2 = m \quad l_1 = l_2 = l$$

$$m\ddot{x} + \gamma \dot{x} + m\frac{g}{l}x + k(x-y) = F\cos\alpha t$$
$$m\ddot{y} + \gamma \dot{y} + m\frac{g}{l}y + k(y-x) = 0$$

$$(\ddot{x}+\ddot{y}) + \frac{\gamma}{m}(\dot{x}+\dot{y}) + \frac{g}{l}(x+y) = \frac{F}{m}\cos\alpha t$$

$$(\ddot{x}-\ddot{y}) + \frac{\gamma}{m}(\dot{x}-\dot{y}) + \frac{g}{l}(x-y) + \frac{2k}{m}(x-y) = \frac{F}{m}\cos\alpha t$$

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \frac{g}{l}q_1 = \frac{F}{m}\cos\alpha t$$

$$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \left(\frac{g}{l} + \frac{2k}{m}\right)q_2 = \frac{F}{m}\cos\alpha t$$

$$q_1 = x + y$$
$$q_2 = x - y$$

CF already done, PI …

- $$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \frac{g}{l}q_1 = \frac{F}{m}\cos\alpha t = \mathrm{Re}\left[\frac{F}{m}\exp(i\alpha t)\right], \qquad q_1 = x + y$$

PI $$q_1 = \mathrm{Re}\left[A_1 \exp(i\alpha t)\right]$$

$$\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l}\right)A_1 = F/m$$

$$A_1 = \frac{(F/m)\exp(i\phi_1)}{\left(\left(\frac{g}{l} - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}} = \frac{(F/m)\exp(i\phi_1)}{\left(\left(\omega_1^2 - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}$$

where $\tan\phi_1 = \dfrac{-\alpha(\gamma/m)}{(\omega_1^2 - \alpha^2)}$ and $\omega_1 = \sqrt{\dfrac{g}{l}}$ the undamped normal mode frequency

$$q_1 = \frac{(F/m)}{\left(\left(\omega_1^2 - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}\cos(\alpha t + \phi_1)$$

- $$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \left(\frac{g}{l} + \frac{2k}{m}\right)q_2 = \frac{F}{m}\cos\alpha t \qquad q_2 = x - y$$

PI

$$q_2 = \mathrm{Re}\left[A_2 \exp\left(i\alpha t\right)\right]$$

$$A_2 = \frac{\left(F/m\right)\exp\left(i\phi_2\right)}{\left(\left(\frac{g}{l} + \frac{2k}{m} - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}} = \frac{\left(F/m\right)\exp\left(i\phi_2\right)}{\left(\left(\omega_2^2 - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}, \qquad \tan\phi_2 = \frac{-\alpha\left(\gamma/m\right)}{\left(\omega_2^2 - \alpha^2\right)}$$

$$q_2 = \frac{\left(F/m\right)}{\left(\left(\omega_2^2 - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}\cos\left(\alpha t + \phi_2\right)$$

$$q_1 = \frac{\left(F/m\right)}{\left(\left(\omega_1^2 - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}\cos\left(\alpha t + \phi_1\right)$$

Finally $x = \frac{1}{2}(q_1 + q_2), \quad y = \frac{1}{2}(q_1 - q_2)$

$$c.f.\begin{pmatrix} x \\ y \end{pmatrix} = \left(-\frac{F}{2m}\cos\alpha t\right)\begin{pmatrix} \frac{1}{\left(\alpha^2 - \omega_1^2\right)} + \frac{1}{\left(\alpha^2 - \omega_2^2\right)} \\ \frac{1}{\left(\alpha^2 - \omega_1^2\right)} - \frac{1}{\left(\alpha^2 - \omega_2^2\right)} \end{pmatrix}, \quad \gamma = 0$$

$$\begin{pmatrix} \frac{d^2}{dt^2}+\frac{\gamma}{m_1}\frac{d}{dt}+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2}+\frac{\gamma}{m_2}\frac{d}{dt}+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\omega t}\right)$$

The case $m_1 = m_2 = m, \quad l_1 \neq l_2$, no damping, no driving force.

Eigenvalue equation

$$\begin{vmatrix} -\omega^2+\left(\frac{g}{l_1}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+\left(\frac{g}{l_2}+\frac{k}{m}\right) \end{vmatrix} = 0$$

$A = g/l_1 + k/m = \beta_1^2 + k/m$
$B = -k/m$
$C = g/l_2 + k/m = \beta_2^2 + k/m$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left\{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t}\right\}$$

$$\omega_{1,2}^2 = \frac{1}{2}\left[\left(\beta_1^2+\beta_2^2\right)+2k/m \pm \sqrt{\left(\beta_1^2-\beta_2^2\right)^2+\left(2k/m\right)^2}\right]$$

$$\frac{x_0}{y_0} = -\frac{m}{2k}\left[\left(\beta_1^2-\beta_2^2\right) \pm \sqrt{\left(\beta_1^2-\beta_2^2\right)^2+\left(2k/m\right)^2}\right]$$

$$\left(\frac{y_0}{x_0}\right)_1 = -1/\left(\frac{y_0}{x_0}\right)_2 \equiv r$$

$$\left(\frac{y_0}{x_0}\right)_1 = -1 \Big/ \left(\frac{y_0}{x_0}\right)_2 \equiv r$$

$$(x_0)_1 = De^{i\delta_1}, \quad (x_0)_2 = De^{i\delta_2}$$

$$\mathbf{x}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ r \end{pmatrix} D\cos(\omega_1 t + \delta_1) + \begin{pmatrix} -r \\ 1 \end{pmatrix} G\cos(\omega_2 t + \delta_2)$$

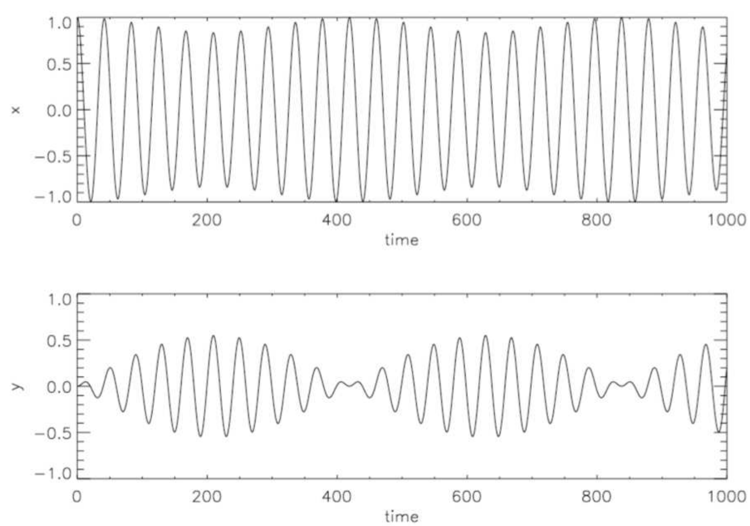
Initial conditions : $\mathbf{x}(0) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad \dot{\mathbf{x}} = 0$

$$\bar{\omega} = \frac{\omega_1 + \omega_2}{2}, \quad \Delta\omega = \omega_1 - \omega_2$$

$$x(t) = a\left[\cos\omega_1 t + r^2\cos\omega_2 t\right] / \left(1 + r^2\right)$$
$$y(t) = ar\left[\cos\omega_1 t - \cos\omega_2 t\right] / \left(1 + r^2\right)$$

$$\Rightarrow$$

$$x(t) = a\cos(\bar{\omega}t)\cos(\Delta\omega t/2) - a\left(\frac{1 - r^2}{1 + r^2}\right)\sin(\bar{\omega}t)\sin(\Delta\omega t/2)$$
$$y(t) = 2ar\sin(\bar{\omega}t)\sin(\Delta\omega t/2) / \left(1 + r^2\right)$$



$$\left(\frac{1 - r^2}{1 + r^2}\right) a \le |x| \le a$$

$$0 \le y \le \left(\frac{2r}{1 + r^2}\right) a$$

Diagrammatic Representation of Normal Modes

$$\mathbf{v}_{1,2} = \left(x_0\mathbf{i} + y_0\mathbf{j}\right) / \sqrt{x_0^2 + y_0^2}$$

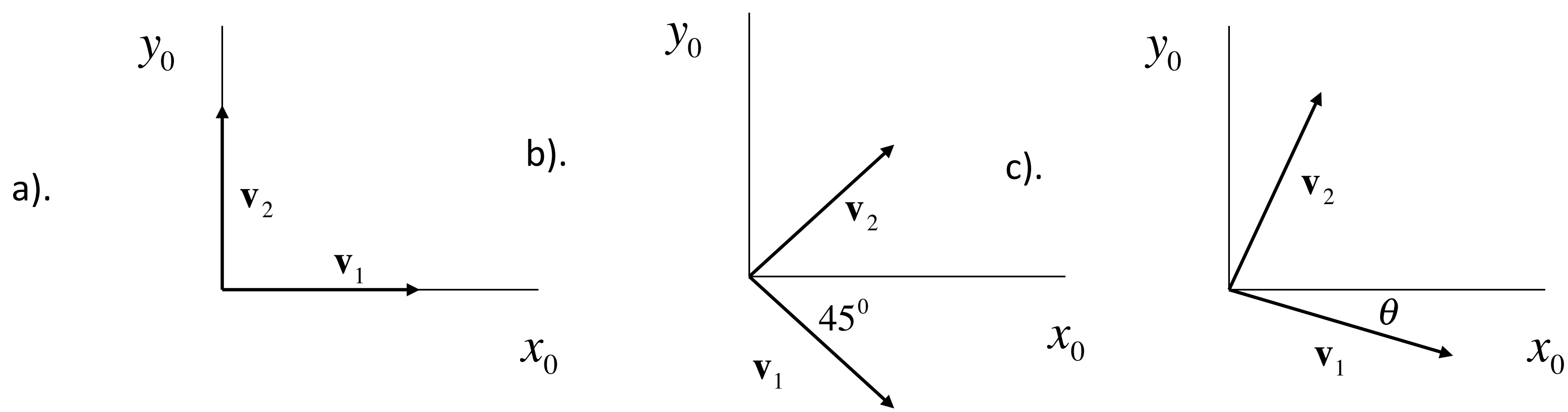
$$\left(\frac{x_0}{y_0}\right)_{1,2} = -\frac{m}{2k}\left[\left(\beta_1^2 - \beta_2^2\right) \pm \sqrt{\left(\beta_1^2 - \beta_2^2\right)^2 + \left(2k/m\right)^2}\right]$$

(a) For $k/m \to 0$ $x_0/y_0 \to$ $-\infty$ or 0

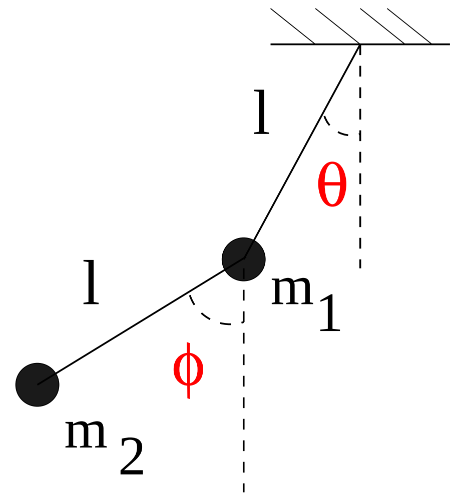
(b) For $k/m \to \infty$ $x_0/y_0 \to$ -1 or 1

(c) Intermediate k/m

$$\frac{y_0}{x_0} = \tan\theta = \frac{-2k/m}{\left(\beta_1^2 - \beta_2^2\right) \pm \sqrt{\left(\beta_1^2 - \beta_2^2\right)^2 + \left(2k/m\right)^2}}$$



Double pendulum



small angle approximation

$\theta\ ,\ \phi << 1$

$$m_1\, l\, \ddot{\theta} = -m_1\, g\, \theta + m_2\, g\, (\phi - \theta)$$

$$m_2\, l\, (\ddot{\theta} + \ddot{\phi}) = -m_2\, g\, \phi$$

$$\frac{d^2}{dt^2}\begin{pmatrix}\theta\\ \phi\end{pmatrix} = -\frac{g}{l}\begin{pmatrix}(m_1+m_2)/m_1 & -m_2/m_1\\ -(m_1+m_2)/m_1 & (m_1+m_2)/m_1\end{pmatrix}\begin{pmatrix}\theta\\ \phi\end{pmatrix}$$

Normal modes:

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \phi_0 \end{pmatrix} e^{i\omega t}$$

$$\Longrightarrow \begin{pmatrix} -\omega^2 + (g/l)(m_1+m_2)/m_1 & -(g/l)(m_2/m_1) \\ -(g/l)(m_1+m_2)/m_1 & -\omega^2 + (g/l)(m_1+m_2)/m_1 \end{pmatrix} \begin{pmatrix} \theta_0 \\ \phi_0 \end{pmatrix} = 0$$

$$\det \begin{pmatrix} -\omega^2 + (g/l)(m_1+m_2)/m_1 & -(g/l)(m_2/m_1) \\ -(g/l)(m_1+m_2)/m_1 & -\omega^2 + (g/l)(m_1+m_2)/m_1 \end{pmatrix} = 0$$

$$\Rightarrow \omega^2 - (g/l)(m_1+m_2)/m_1 = \pm\, (g/l)\sqrt{m_2(m_1+m_2)}/m_1$$

i.e.,

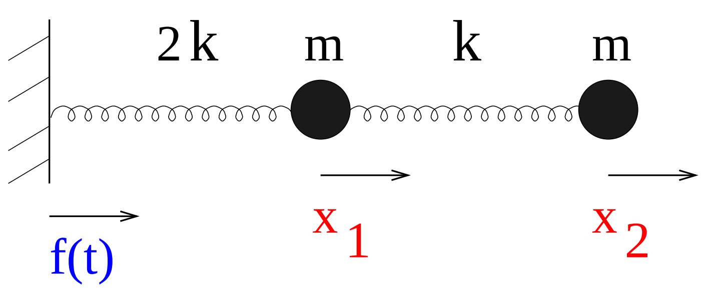
$$\omega_\pm = \sqrt{\frac{g}{l}\ \frac{1}{1 \mp \sqrt{m_2/(m_1+m_2)}}} \qquad \text{normal frequencies}$$

Example [CP4 June 2006]

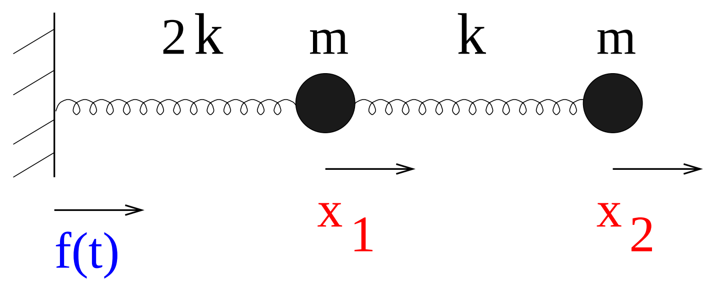
12. Two identical masses $m_1 = m_2 = m$ are connected by a massless spring with spring constant k. Mass m_1 is attached to a support by another massless spring with spring constant $2k$. The masses and springs lie along the horizontal x-axis on a smooth surface. The masses and the support are allowed to move along the x-axis only. The displacement of the support in the x-direction at time t is given by $f(t)$ and is externally controlled. Write down a system of differential equations describing the evolution of the displacements x_1 and x_2 of the masses from their equilibrium positions. [5]

Determine the frequencies of the normal modes and their amplitude ratios. [8]

The displacement of the support is given by $f(t) = A\sin(\omega t)$ with $\omega^2 = k/m$ and constant amplitude A. Find expressions for $x_1(t)$ and $x_2(t)$ assuming that any transients have been damped out by a small, otherwise negligible, damping term. [7]



◊ Coupled oscillators with a driving term:



$$m\ddot{x}_2 = -k(x_2 - x_1)$$

$$m\ddot{x}_1 = -2k(x_1 - f(t)) - k(x_1 - x_2)$$

- Homogeneous case ($f = 0$):

$$\longrightarrow \quad \text{normal frequencies } \omega_1 = \sqrt{\frac{k}{m}(2+\sqrt{2})} \; , \quad \omega_2 = \sqrt{\frac{k}{m}(2-\sqrt{2})}$$

$$\text{amplitude ratios}: \quad (X_2/X_1)_{NM1} = 1 - \sqrt{2} \; , \quad (X_2/X_1)_{NM2} = 1 + \sqrt{2}$$

- Driving term $f(t) = A\sin\omega t$, $\omega = \sqrt{k/m}$:

$$x_1(t) = C_1 \mathrm{Im}\ e^{i\omega t} \quad , \qquad x_2(t) = C_2 \mathrm{Im}\ e^{i\omega t}$$

$$\Longrightarrow \qquad -\omega^2 \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \omega^2 \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \omega^2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} A$$

$$\text{Thus}: \quad -C_1 = -3C_1 + C_2 + 2A$$

$$-C_2 = C_1 - C_2$$

$$\Longrightarrow \qquad C_1 = 0, C_2 = -2A$$

$$x_1(t) = 0 \quad , \qquad x_2(t) = -2A\sin\omega t$$