# Lecture 3

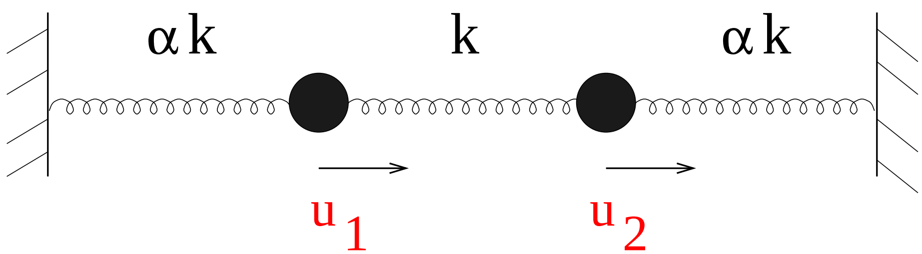
## Normal Modes

♢ Spring-mass systems

♢ Coupled oscillators with damping and forcing terms

# SPRING-MASS SYSTEMS

- Two masses $m$ moving on a straight line without friction under the action of three springs:




$$m\ddot{u}_1 = -\alpha k u_1 - k(u_1 - u_2)$$

$$m\ddot{u}_2 = -\alpha k u_2 + k(u_1 - u_2)$$

- Determine normal modes of the system.
- Express the energy in terms of normal modes.
- Find solution for initial conditions $u_1 = u_0, \;\; u_2 = 0, \;\; \dot{u}_1 = 0, \;\; \dot{u}_2 = 0.$

$$m\ddot{u}_1 = -\alpha k u_1 - k(u_1 - u_2)$$

$$m\ddot{u}_2 = -\alpha k u_2 + k(u_1 - u_2)$$

Setting

$$S = \frac{u_1 + u_2}{\sqrt{2}} \quad , \quad D = \frac{u_1 - u_2}{\sqrt{2}}$$

normal coordinates

gives

$$m\ddot{S} + \alpha k S = 0$$

$$m\ddot{D} + (\alpha + 2)kD = 0$$

$$\Rightarrow \ \omega_S = \sqrt{\frac{\alpha k}{m}} \quad , \quad \omega_D = \sqrt{\frac{(\alpha + 2)k}{m}}$$

normal frequencies

Kinetic energy : $K = \frac{1}{2}\, m\, (\dot{u}_1^2 + \dot{u}_2^2) = \frac{1}{2}\, m\, (\dot{S}^2 + \dot{D}^2)$

Potential energy : $$\begin{aligned} V &= \frac{1}{2}\, \alpha k u_1^2 + \frac{1}{2}\, k\, (u_2 - u_1)^2 + \frac{1}{2}\, \alpha k u_2^2 \\ &= \frac{1}{2}\, \alpha k S^2 + \frac{1}{2}\, (\alpha + 2) k D^2 = \frac{1}{2}\, m\omega_S^2 S^2 + \frac{1}{2}\, m\omega_D^2 D^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow\ E &= \underbrace{\frac{1}{2}\, m\dot{S}^2 + \frac{1}{2}\, m\omega_S^2 S^2}_{E_1} + \underbrace{\frac{1}{2}\, m\dot{D}^2 + \frac{1}{2}\, m\omega_D^2 D^2}_{E_2} \\ &= E_1 + E_2 \qquad \textit{sum of the energies of each normal mode} \end{aligned}$$

General solution of the equations of motion:

$$S(t) = C_S \sin(\omega_S t + \varphi_S) \ \ , \ \ D(t) = C_D \sin(\omega_D t + \varphi_D)$$

that is, $u_1(t) = \frac{1}{\sqrt{2}} \ [C_S \sin(\omega_S t + \varphi_S) + C_D \sin(\omega_D t + \varphi_D)]$

$$u_2(t) = \frac{1}{\sqrt{2}} \ [C_S \sin(\omega_S t + \varphi_S) - C_D \sin(\omega_D t + \varphi_D)]$$

Given the initial conditions at time $t = 0$

$$u_1 = u_0 \ \ , \quad u_2 = 0 \ \ , \quad \dot{u}_1 = 0 \ \ , \quad \dot{u}_2 = 0$$

the solution satisfying these conditions is given by

$$u_1(t) = \frac{1}{2} \ u_0 \left( \cos \omega_S t + \cos \omega_D t \right)$$

$$u_2(t) = \frac{1}{2} \ u_0 \left( \cos \omega_S t - \cos \omega_D t \right)$$

## Homework

Three equal masses $m$ are constrained to move on a circle without friction, subject to the action of three springs of elastic constant $k$ connecting the three masses pairwise to each other. The equations of motion are given by

$$m\ddot{u}_1 = -k(u_1 - u_2) + k(u_3 - u_1) \ ,$$

$$m\ddot{u}_2 = -k(u_2 - u_3) + k(u_1 - u_2) \ ,$$

$$m\ddot{u}_3 = -k(u_3 - u_1) + k(u_2 - u_3) \ .$$

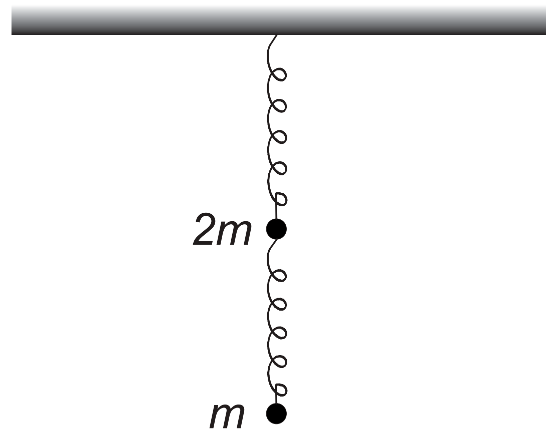
Determine normal frequencies and normal coordinates of the system.

Answ.: Normal frequencies $\omega_1 = 0$, $\omega_2 = \omega_3 = \sqrt{\frac{3k}{m}}$.
Normal coordinates $q_1 = \frac{1}{\sqrt{3}}\ (u_1 + u_2 + u_3)$, $q_2 = \frac{1}{\sqrt{6}}\ (u_1 - 2u_2 + u_3)$, $q_3 = \frac{1}{\sqrt{2}}\ (u_1 - u_2)$.

**8.** Two massless springs each have spring constant $k$. Masses $2m$ and $m$ are attached as shown in the figure.

Example

[CP4 Paper June 2008]




The masses make small vertical oscillations about their equilibirium positions. Show that the respective displacements $x$ and $y$ of the masses $2m$ and $m$ satisfy the coupled differential equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{k}{2m}(y-2x)$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{k}{m}(x-y)$$

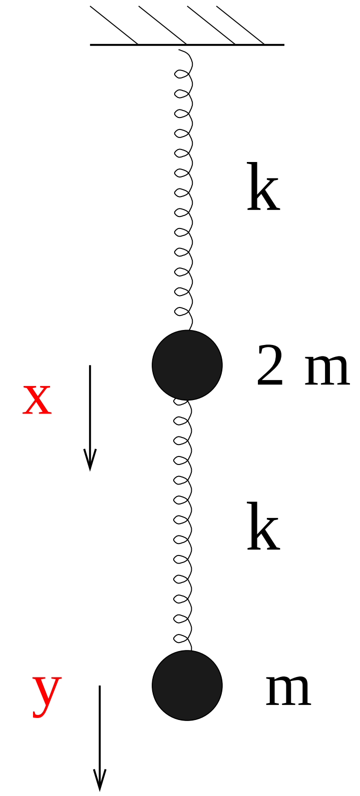
and explain why there is no term involving the acceleration due to gravity. [7]

Find expressions for the normal frequencies for small oscillations of the masses. [7]

Find the ratio of the amplitudes for each normal mode. [6]

$$m\,\ddot{y} = -\,k\,(\,y - x\,)$$

$$2\,m\,\ddot{x} = -\,k\,x - k\,(\,x - y\,)$$




*g does not appear because x and y are displacements from equilibrium (gravity will determine shift mg/k of the zero)*

$$\ddot{y} = (k/m)(x - y)$$

$$\ddot{x} = [k/(2m)](y - 2x)$$

# Matrix method

$$\text{Ansatz} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t} \longrightarrow \begin{pmatrix} -\omega^2 + k/m & -k/2m \\ -k/m & -\omega^2 + k/m \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\begin{vmatrix} -\omega^2 + k/m & -k/2m \\ -k/m & -\omega^2 + k/m \end{vmatrix} = (-\omega^2 + k/m)^2 - (k/m)^2/2 = 0$$

$$\Rightarrow \omega^2 - \frac{k}{m} = \pm \frac{1}{\sqrt{2}} \frac{k}{m}$$

i.e.,

$$\omega^2 = \frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right) \qquad \text{normal frequencies}$$

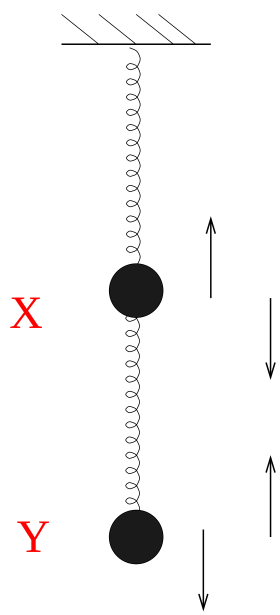
- Normal mode 1: $\omega^2 = \omega_1^2 = (k/m)\left(1 + 1/\sqrt{2}\right)$

$$(-\omega_1^2 + k/m)X = [k/(2m)]Y \implies -X/\sqrt{2} = Y/2 \quad i.e., \qquad X/Y = -1/\sqrt{2}$$

- Normal mode 2: $\omega^2 = \omega_2^2 = (k/m)\left(1 - 1/\sqrt{2}\right)$

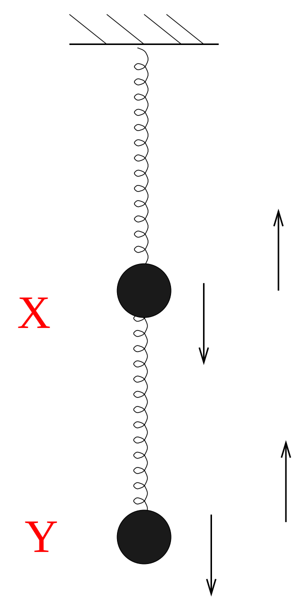
$$(-\omega_2^2 + k/m)X = [k/(2m)]Y \implies X/\sqrt{2} = Y/2 \quad i.e., \qquad X/Y = 1/\sqrt{2}$$

NM 1



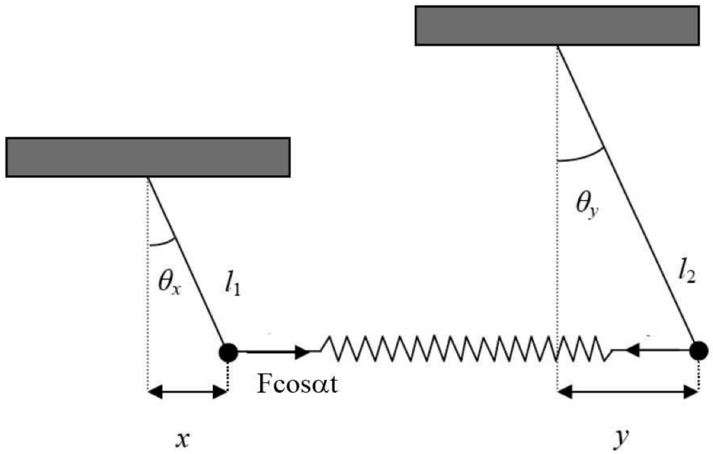

*move against*

NM 2




*move together*

# The damped driven pendulum




$$m_1\ddot{x} = -\gamma\dot{x} - m_1 g x / l_1 + k\left(y - x\right) + F\cos\alpha t$$
$$m_2\ddot{y} = -\gamma\dot{y} - m_2 g y / l_2 - k\left(y - x\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{\gamma}{m_1}\frac{d}{dt} + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2} + \frac{\gamma}{m_2}\frac{d}{dt} + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{\gamma}{m_1}\frac{d}{dt} + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2} + \frac{\gamma}{m_2}\frac{d}{dt} + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{Re}\left(e^{i\alpha t}\right)$$

CF

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}\right)$$

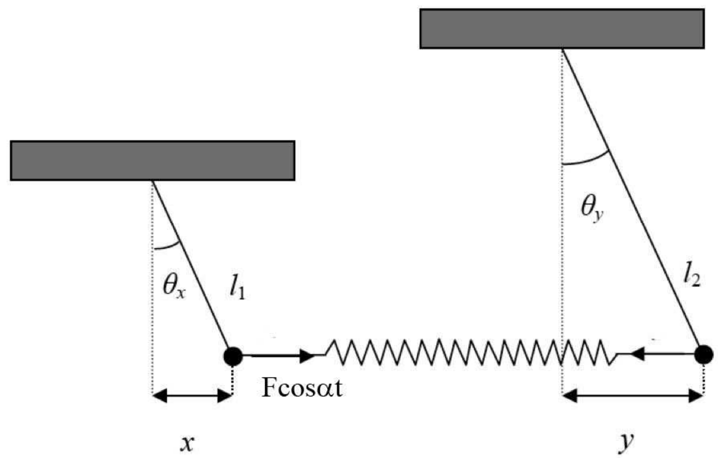
$$\begin{pmatrix} -\omega^2 + i\frac{\gamma}{m_1}\omega + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2 + i\frac{\gamma}{m_2}\omega + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (1)$$

$$\begin{vmatrix} -\omega^2 + i\frac{\gamma}{m_1}\omega + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2 + i\frac{\gamma}{m_2}\omega + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{vmatrix} = 0$$

Eigenvalue eq.

Substitute (complex) eigenvalues in (1) to obtain eigenvectors

The damped driven pendulum - the Particular Integral




$$m_1\ddot{x} = -\gamma\dot{x} - m_1 g x / l_1 + k\left(y - x\right) + F\cos\alpha t$$
$$m_2\ddot{y} = -\gamma\dot{y} - m_2 g y / l_2 - k\left(y - x\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{\gamma}{m_1}\frac{d}{dt} + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2} + \frac{\gamma}{m_2}\frac{d}{dt} + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2}+\frac{\gamma}{m_1}\frac{d}{dt}+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2}+\frac{\gamma}{m_2}\frac{d}{dt}+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

PI

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\begin{pmatrix} P \\ Q \end{pmatrix} e^{i\alpha t}\right)$$

$$\begin{pmatrix} -\alpha^2+i\frac{\gamma}{m_1}\alpha+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2+i\frac{\gamma}{m_2}\alpha+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} \equiv \mathbf{MP} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{P} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} = \mathbf{M}^{-1}\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} -\alpha^2+i\frac{\gamma}{m_1}\alpha+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2+i\frac{\gamma}{m_2}\alpha+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\mathbf{M}^{-1}\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\alpha t}\right)$$

Simple case $m_1 = m_2 = m \quad l_1 = l_2 = l$

$$\begin{vmatrix} -\omega^2 - i\dfrac{\gamma}{m}\omega + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) & -\dfrac{k}{m} \\ -\dfrac{k}{m} & -\omega^2 - i\dfrac{\gamma}{m}\omega + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) \end{vmatrix} = 0$$

$$-\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}\right) = 0 \qquad \text{or} \qquad -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{2k}{m}\right) = 0$$

$$\bar{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2}$$ Eigenvalues

$$\omega_1^2 = \frac{g}{l} \quad \text{or} \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

$\gamma = 0$ eigenvalues (c.f. previous result)