Lecture 2

Normal Modes

$\diamondsuit$ Coupled pendula (recap):

- 2 linear ODEs in $x(t), y(t) \longrightarrow$ 2 *normal frequencies* ω_1, ω_2 at which system can oscillate as a whole.

$\Rightarrow x + y$ and $x - y$ oscillate *independently* at frequencies ω_1 and ω_2 (*normal modes*)

- any generic motion of the system is linear superposition of normal modes : GS $=$ c_1 NM1 $+$ c_2 NM2

Coupled pendula

Normal modes (Matrix method) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x) \qquad x + y = 2A_1 \cos(\omega_1 t + \phi_1)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x) \qquad x - y = 2A_2 \cos(\omega_2 t + \phi_2)$$

Solution II - Decoupling method : (simpler in this case)

$$m(\ddot{x} + \ddot{y}) = -m\frac{g}{l}(x+y)$$

$$\ddot{q}_1 = -\omega_1^2 q_1, \quad q_1 = (x+y), \quad \omega_1^2 = \frac{g}{l}$$

$$q_1 = A_1 \cos(\omega_1 t + \phi_1)$$

centre of mass motion

normal coordinates

$$m(\ddot{x} - \ddot{y}) = (-m\frac{g}{l} - 2k)(x-y)$$

$$\ddot{q}_2 = -\omega_2^2 q_2, \quad q_2 = (x-y), \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

$$q_2 = A_2 \cos(\omega_2 t + \phi_2)$$

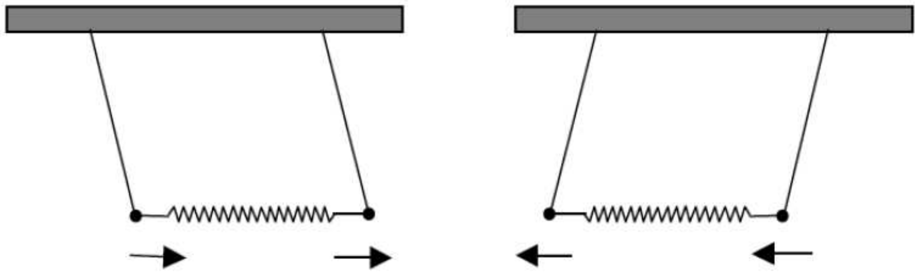
relative motion

$$m(\ddot{x}+\ddot{y}) = -m\frac{g}{l}(x+y)$$

$$q_1 = A_1 \cos\left(\omega_1 t + \phi_1\right)$$

$$\ddot{q}_1 = -\omega_1^2 q_1, \quad q_1 = \left(x+y\right), \quad \omega_1^2 = \frac{g}{l}$$

centre of mass motion

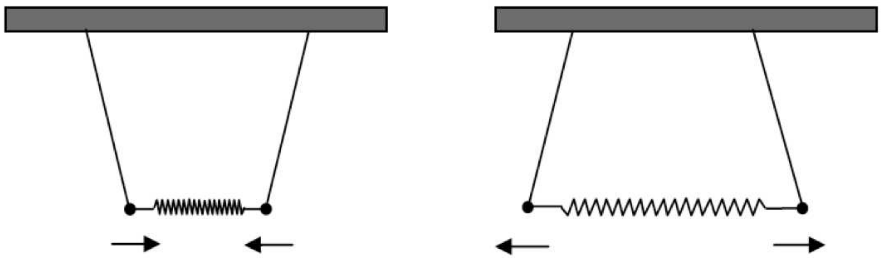


$$m(\ddot{x}-\ddot{y}) = (-m\frac{g}{l} - 2k)(x-y)$$

$$q_2 = A_2 \cos\left(\omega_2 t + \phi_2\right)$$

$$\ddot{q}_2 = -\omega_2^2 q_2, \quad q_2 = \left(x-y\right), \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

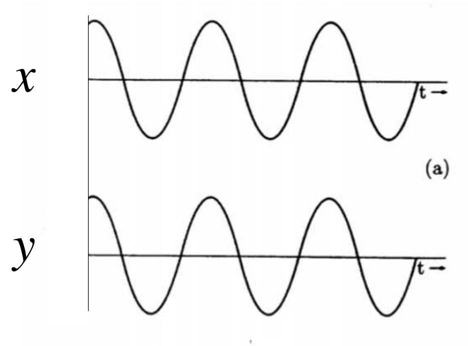
relative motion



General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

Constants determined by initial condtions



Ex 1 $\quad x(0) = y(0) = a \quad \dot{x}(0) = \dot{y}(0) = 0$

$$\begin{aligned} &A_1 \cos(\phi_1) + A_2 \cos(\phi_2) = a \\ &A_1 \cos(\phi_1) - A_2 \cos(\phi_2) = a \\ &A_1\omega_1 \sin(\phi_1) + A_2\omega_2 \sin(\phi_2) = 0 \\ &A_1\omega_1 \sin(\phi_1) - A_2\omega_2 \sin(\phi_2) = 0 \end{aligned}$$

$$\Rightarrow A_2 = 0, A_1 = a, \phi_1 = 0$$

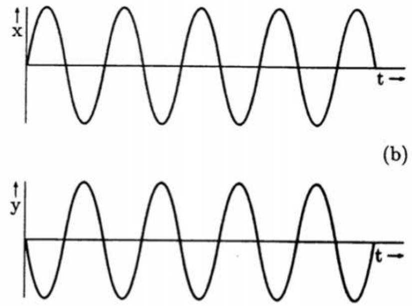
$$\begin{pmatrix} x \\ y \end{pmatrix} = a\cos(\omega_1 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t > 0$$

1st Normal mode excitation

General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos\left(\omega_1 t + \phi_1\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos\left(\omega_2 t + \phi_2\right)$$

Constants determined by initial condtions



Ex2 $\quad x(0) = y(0) = 0 \quad \dot{x}(0) = -\mathrm{v}, \quad \dot{y}(0) = \mathrm{v}$

$$\begin{aligned} A_1 \cos(\phi_1) + A_2 \cos(\phi_2) &= 0 \\ A_1 \cos(\phi_1) - A_2 \cos(\phi_2) &= 0 \\ A_1\omega_1 \sin(\phi_1) + A_2\omega_2 \sin(\phi_2) &= \text{-v} \\ A_1\omega_1 \sin(\phi_1) - A_2\omega_2 \sin(\phi_2) &= \mathrm{v} \end{aligned}$$

$$\Rightarrow A_1 = 0, \quad \phi_2 = \pi / 2, \quad A_2 = -\frac{\mathrm{v}}{\omega_2}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\mathrm{v}}{\omega_2} \sin\left(\omega_1 t\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t > 0$$

2nd Normal mode excitation

General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos\left(\omega_1 t + \phi_1\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos\left(\omega_2 t + \phi_2\right)$$

Constants determined by initial conditions

Ex3 $\quad x(0) = a \quad y(0) = 0 \quad \dot{x}(0) = \dot{y}(0) = 0$

$$\begin{aligned} A_1 \cos(\phi_1) + A_2 \cos(\phi_2) &= a \\ A_1 \cos(\phi_1) - A_2 \cos(\phi_2) &= 0 \\ A_1\omega_1 \sin(\phi_1) + A_2\omega_2 \sin(\phi_2) &= 0 \\ A_1\omega_1 \sin(\phi_1) - A_2\omega_2 \sin(\phi_2) &= 0 \end{aligned} \quad \Rightarrow \quad \phi_1 = \phi_2 = 0, \quad A_1 = A_2 = \frac{a}{2}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{a}{2}\cos\left(\omega_1 t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{a}{2}\cos\left(\omega_2 t\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

both modes excited

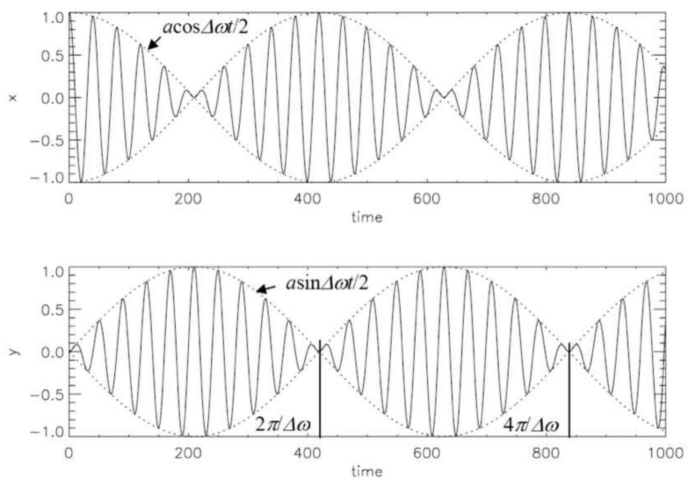
$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{a}{2}\cos(\omega_1 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{a}{2}\cos(\omega_2 t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

both modes excited

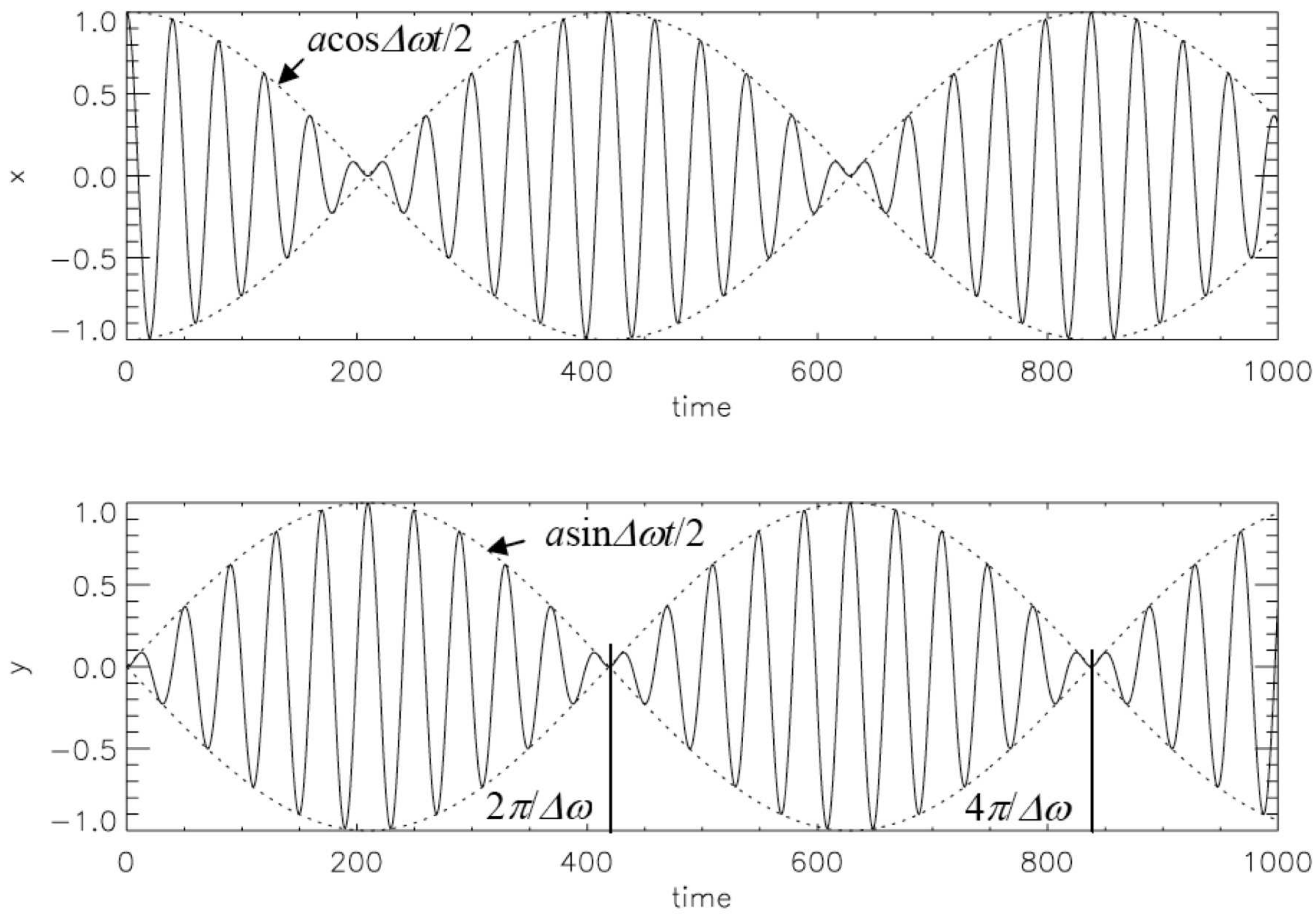
$$x = a\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right) \quad = a\cos(\overline{\omega}t)\cos(\Delta\omega t/2)$$

$$y = a\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right) \quad = a\sin(\overline{\omega}t)\sin(\Delta\omega t/2)$$

where $\overline{\omega} = \dfrac{\omega_1+\omega_2}{2}$ and $\Delta\omega = \omega_1 - \omega_2$

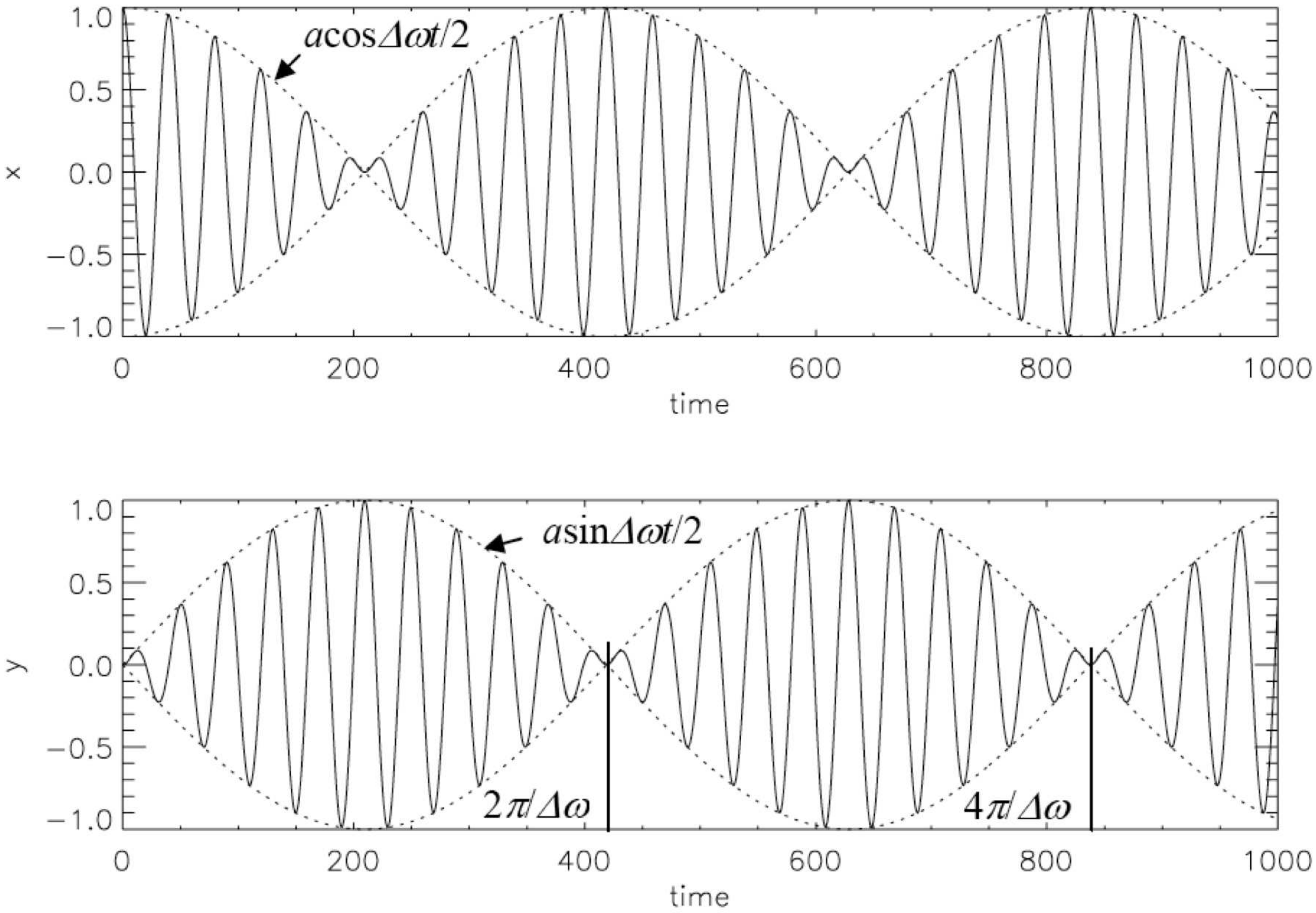


$\Delta\omega << \overline{\omega}$



$$x = a\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right) \qquad = a\cos(\overline{\omega}t)\cos(\Delta\omega t/2)$$

$$y = a\sin\left(\frac{\omega_1 + \omega_2}{2}t\right)\sin\left(\frac{\omega_1 - \omega_2}{2}t\right) \qquad = a\sin(\overline{\omega}t)\sin(\Delta\omega t/2)$$



Note transfer of energy between the two pendula

“Beats”

Energy of motion

P.E. V + K.E. K (Potential energy+Kinetic energy)

$$m\ddot{x} = -mg\frac{x}{l} + k(y - x) \quad = -\frac{\partial V}{\partial x}$$

Conservative force

Energy of motion

P.E. V + K.E. K

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x) = -\frac{\partial V}{\partial x}$$

$$V(x,y) = mg\frac{x^2}{2l} + \frac{1}{2}kx^2 - kxy + f(y)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x) = -\frac{\partial V}{\partial y} = kx - \frac{df}{dy}$$

$$\Rightarrow f(y) = mg\frac{y^2}{2l} + \frac{1}{2}ky^2 + C$$

$$V(x,y) = \frac{1}{2}m\left(\frac{g}{l} + \frac{k}{m}\right)(x^2 + y^2) - kxy + C$$

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

In normal coordinates:

$$q_1 = \frac{1}{\sqrt{2}}(x+y), \quad q_2 = \frac{1}{\sqrt{2}}(x-y)$$

...with $\frac{1}{\sqrt{2}}$ q_1, q_2 rotated axis w.r.t. x,y

$$x = \frac{1}{\sqrt{2}}(q_1+q_2), \quad y = \frac{1}{\sqrt{2}}(q_1-q_2)$$

Energy of motion

P.E. V + K.E. K

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x) = -\frac{\partial V}{\partial x}$$

$$V(x,y) = mg\frac{x^2}{2l} + \frac{1}{2}kx^2 - kxy + f(y)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x) = -\frac{\partial V}{\partial y} = kx - \frac{df}{dy}$$

$$\Rightarrow f(y) = mg\frac{y^2}{2l} + \frac{1}{2}ky^2 + C$$

$$\boxed{V(x,y) = \frac{1}{2}m\left(\frac{g}{l} + \frac{k}{m}\right)(x^2+y^2) - kxy + C}$$

$$\boxed{K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)}$$

In normal coordinates:

$$q_1 = \frac{1}{\sqrt{2}}(x+y), \quad q_2 = \frac{1}{\sqrt{2}}(x-y)$$

$$x = \frac{1}{\sqrt{2}}(q_1+q_2), \quad y = \frac{1}{\sqrt{2}}(q_1-q_2)$$

$$\boxed{\begin{aligned} V &= \frac{1}{2}m\frac{g}{l}q_1^2 + \frac{1}{2}m\left(\frac{g}{l} + \frac{2k}{m}\right)q_2^2 \\ &= \frac{1}{2}m\omega_1^2 q_1^2 + \frac{1}{2}m\omega_2^2 q_2^2 \end{aligned}}$$

$$\boxed{K = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right)}$$

$$V = \frac{1}{2} m \frac{g}{l} q_1^2 + \frac{1}{2} m \left(\frac{g}{l} + \frac{2k}{m} \right) q_2^2$$
$$= \frac{1}{2} m\omega_1^2 q_1^2 + \frac{1}{2} m\omega_2^2 q_2^2$$

$$\ddot{q}_1 = -\omega_1^2 q_1, \quad q_1 = \frac{1}{\sqrt{2}}(x+y), \quad \omega_1^2 = \frac{g}{l}$$

$$K = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2 \right)$$

$$\ddot{q}_2 = -\omega_2^2 q_2, \quad q_2 = \frac{1}{\sqrt{2}}(x-y), \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

$$E = \left(\frac{1}{2} m\omega_1^2 q_1^2 + \frac{1}{2} m\dot{q}_1^2 \right) + \left(\frac{1}{2} m\omega_2^2 q_2^2 + \frac{1}{2} m\dot{q}_2^2 \right)$$

$$= E_1 + E_2$$

Total energy = sum of energies of the normal modes

Parseval’s theorem