Normal Modes, Wave Motion and the Wave Equation

Hilary Term 2012

Lecturer: F Hautmann

- Part A: Normal modes ($\sim$ 4 lectures)
- Part B: Waves ($\sim$ 8 lectures)

- printed lecture notes
- slides will be posted on lecture webpage: http://www-thphys.physics.ox.ac.uk/people/FrancescoHautmann/Cp4p/
- suggested problem sheets also on webpage

References

- Textbooks covering aspects of this course include

[1] French: Vibrations and Waves, MIT Introductory Physics Series

[2] Coulson and Jeffrey: Waves, Longman

A. Normal modes

1. Systems of linear ordinary differential equations

2. Solution by normal coordinates and normal modes

3. Applications to coupled oscillators

B. Waves

- ▷ Partial differential equations (PDEs).
- ▷ The wave equation.
- ▷ Traveling waves. Stationary waves.
- ▷ Dispersion. Phase and group velocities.
- ▷ Reflection and transmission of waves.

Introduction to Normal Modes

• Consider a physical system with N degrees of freedom whose dynamics is described by a set of coupled linear ODEs.

• To determine the *normal modes* of the system means to find a set of N coordinates (*normal coordinates*) describing the system which evolve *independently* like N harmonic oscillators.

• The frequencies of such harmonic motion are the *normal frequencies* of the system.

▷ normal modes describe "collective" motion of the system

▷ general solution expressible as linear superposition of normal modes

SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

- more than 1 unknown function: $y_1(x), y_2(x), \ldots, y_n(x)$
- set of ODEs that couple $y_1, \ldots, y_n$

▷ physical applications: systems with more than 1 degree of freedom. dynamics couples differential equations for different variables.

Example. System of first-order differential equations:

$$y_1' = F_1(x, y_1, y_2, \ldots, y_n)$$

$$y_2' = F_2(x, y_1, y_2, \ldots, y_n)$$

$$\cdots$$

$$y_n' = F_n(x, y_1, y_2, \ldots, y_n)$$

♢ Systems of linear ODEs with constant coefficients can be solved by a generalization of the method seen for single ODE:

General solution $= \mathrm{PI} + \mathrm{CF}$

▷ Complementary function CF by solving *system of auxiliary equations*

▷ Particular integral PI from a *set of trial functions* with functional form as the inhomogeneous terms

Example. Solve

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= 2\sin t + 3\cos t + 5\mathrm{e}^{-t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - y &= 3\cos t - 5\sin t - \mathrm{e}^{-t} \end{aligned} \quad \text{given} \quad \begin{aligned} x(0) &= 2; \quad y(0) = -3 \\ \dot{x}(0) &= 0; \quad \dot{y}(0) = 4 \end{aligned}$$

To find CF

Set $x = X\mathrm{e}^{\alpha t}$, $y = Y\mathrm{e}^{\alpha t}$

$$\Rightarrow \quad \begin{matrix} (\alpha^2+2)X & + & \alpha Y \\ \alpha X & + & (\alpha^2-1)Y \end{matrix} = 0 \quad \Rightarrow \quad \alpha^4 = 2$$

$$\Rightarrow \quad \alpha^2 = \pm\sqrt{2} \quad \Rightarrow \quad \alpha = \pm\beta,\ \pm\mathrm{i}\beta \quad (\beta \equiv 2^{1/4})$$

and $Y/X = -(\alpha^2+2)/\alpha$ so the CF is

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} \beta \\ 2+\sqrt{2} \end{pmatrix} \mathrm{e}^{-\beta t} + X_b \begin{pmatrix} -\beta \\ 2+\sqrt{2} \end{pmatrix} \mathrm{e}^{\beta t} + X_c \begin{pmatrix} \mathrm{i}\beta \\ 2-\sqrt{2} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\beta t} + X_d \begin{pmatrix} -\mathrm{i}\beta \\ 2-\sqrt{2} \end{pmatrix} \mathrm{e}^{\mathrm{i}\beta t}$$

To Find PI

Set $(x, y) = (X, Y)\mathrm{e}^{-t} \quad \Rightarrow$

$$\begin{aligned} X - Y + 2X &= 5 \\ -X + Y - Y &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X &= 1 \\ Y &= -2 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-t}$$

Have $2\sin t + 3\cos t = \Re\mathrm{e}(\sqrt{13}\mathrm{e}^{\mathrm{i}(t+\phi)})$, where $\cos\phi = 3/\sqrt{13}$, $\sin\phi = -2/\sqrt{13}$.
Similarly $3\cos t - 5\sin t = \Re\mathrm{e}(\sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)})$, where $\cos\psi = 3/\sqrt{34}$, $\sin\psi = 5/\sqrt{34}$
Set $(x, y) = \Re\mathrm{e}[(X, Y)\mathrm{e}^{\mathrm{i}t}]$ and require

$$\begin{aligned} -X + \mathrm{i}Y + 2X = X + \mathrm{i}Y &= \sqrt{13}\mathrm{e}^{\mathrm{i}\phi} \\ \mathrm{i}X - Y - Y = \mathrm{i}X - 2Y &= \sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \end{aligned} \quad \Rightarrow \quad \begin{aligned} -\mathrm{i}Y &= \sqrt{13}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}\sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \\ \mathrm{i}X &= 2\mathrm{i}\sqrt{13}\mathrm{e}^{\mathrm{i}\phi} - \sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \end{aligned}$$

so

$$\begin{aligned} x &= \Re\mathrm{e}(2\sqrt{13}e^{\mathrm{i}(t+\phi)} + \mathrm{i}\sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)}) \\ &= 2\sqrt{13}(\cos\phi\cos t - \sin\phi\sin t) - \sqrt{34}(\sin\psi\cos t + \cos\psi\sin t) \\ &= 2[3\cos t + 2\sin t] - 5\cos t - 3\sin t \\ &= \cos t + \sin t \end{aligned}$$

Similarly

$$\begin{aligned} y &= \Re\mathrm{e}(\sqrt{13}\mathrm{i}\mathrm{e}^{\mathrm{i}(t+\phi)} - \sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)}) \\ &= \sqrt{13}(-\sin\phi\cos t - \cos\phi\sin t) - \sqrt{34}(\cos\psi\cos t - \sin\psi\sin t) \\ &= 2\cos t - 3\sin t - 3\cos t + 5\sin t \\ &= -\cos t + 2\sin t. \end{aligned}$$

For the initial-value problem

$$\mathrm{PI}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad ; \quad \dot{\mathrm{PI}}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

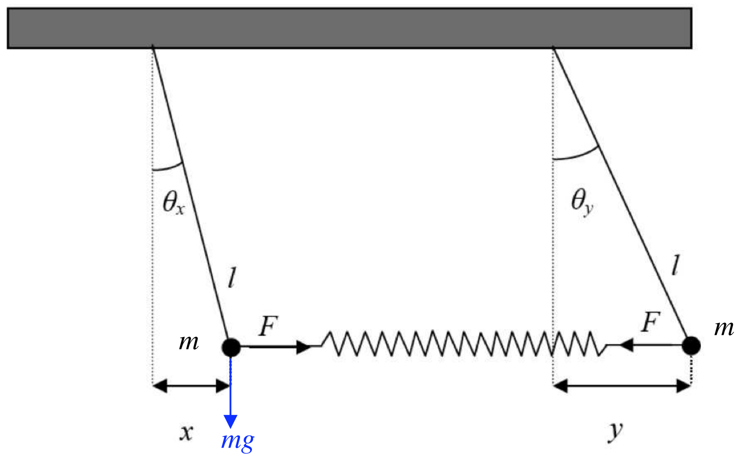
$$\mathrm{CF}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ; \quad \dot{\mathrm{CF}}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore the solution satisfying the initial data is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-t}.$$

Normal Modes

Coupled differential equations - e.g. coupled pendula



$$m\ddot{x} = -mg\frac{x}{l} + k(y-x)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x)$$

Solution I - Matrix method :

$$\begin{pmatrix} \frac{d^2}{dt^2}+\frac{g}{l}+\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2}+\frac{g}{l}+\frac{k}{m} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

CF : Try

$$\begin{pmatrix} x \\ y \end{pmatrix}=\mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix}e^{i\omega t}$$

X,Y (complex) constants

$$\begin{pmatrix} -\omega^2+\frac{g}{l}+\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+\frac{g}{l}+\frac{k}{m} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A.\Psi = 0 \quad \Rightarrow \quad Det[A] = 0$$

Eigenvalue equation

$$\begin{vmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{vmatrix} = 0$$

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \quad \Rightarrow \quad \left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm\left(\frac{k}{m}\right)$$

$$\left(-\omega^2+\frac{g}{l}+\frac{k}{m}\right)=\pm\left(\frac{k}{m}\right)$$

Eigenvalue equation

$$\omega_1^2=\frac{g}{l} \quad \text{or} \quad \omega_2^2=\frac{g}{l}+2\frac{k}{m}$$

Eigenvalues

$$\begin{pmatrix} x \\ y \end{pmatrix}=\mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix}e^{i\omega t} \Longrightarrow \begin{pmatrix} -\omega^2+\frac{g}{l}+\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+\frac{g}{l}+\frac{k}{m} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equation

$$\underline{\omega=\omega_1}: \begin{pmatrix} +\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & +\frac{k}{m} \end{pmatrix}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow X_1=Y_1=A_1e^{i\phi_1}$$

Eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}A_1\cos(\omega_1 t+\phi_1)$$

1st Normal mode

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm\left(\frac{k}{m}\right)$$

Eigenvalue equation

$$\omega_1^2 = \frac{g}{l} \quad \text{or} \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

Eigenvalues

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t} \Rightarrow \begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equation

$$\underline{\omega = \omega_2} \qquad \begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow X_2 = -Y_2 = A_2 e^{i\phi_2}$$

Eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

2nd Normal mode

$$m\ddot{x} = -mg\frac{x}{l} + k(y - x)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y - x)$$

General solution given by a superposition of the two independent (normal mode)solutions:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos\left(\omega_1 t + \phi_1\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos\left(\omega_2 t + \phi_2\right)$$

Normal Modes

$$x + y = 2A_1 \cos\left(\omega_1 t + \phi_1\right)$$

$$x - y = 2A_2 \cos\left(\omega_2 t + \phi_2\right)$$

} N 1D linear differential Equations…N normal modes

Summary

$\diamondsuit$ method of solution for single ODEs
extended to *systems* of coupled differential equations

General solution $= \mathrm{PI} + \mathrm{CF}$

$\diamondsuit$ coupled pendula:

• 2 linear ODEs in $x(t), y(t) \longrightarrow 2$ *normal frequencies* ω_1, ω_2
at which system can oscillate as a whole.

$\Rightarrow x + y$ and $x - y$ oscillate *independently* at frequencies ω_1 and ω_2
(*normal modes*)

• any generic motion of the system is linear superposition of normal modes : GS $=$ c_1 NM1 $+$ c_2 NM2