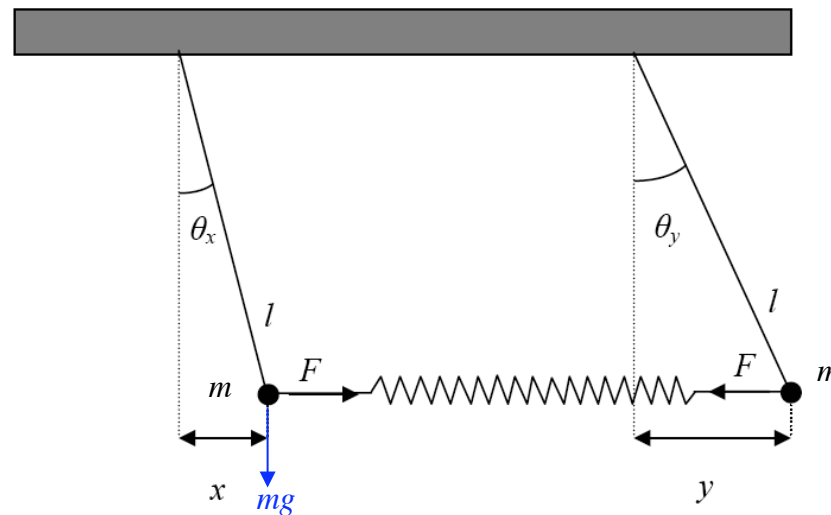


Normal Modes

Coupled differential equations - e.g. coupled pendula



$$m\ddot{x} = -mg \frac{x}{l} + k(y - x)$$

$$m\ddot{y} = -mg \frac{y}{l} - k(y - x)$$

$$\begin{aligned}\ddot{x} + g\frac{x}{l} - \frac{k}{m}x + \frac{k}{m}y &= 0 \\ \ddot{y} + g\frac{y}{l} - \frac{k}{m}y + \frac{k}{m}x &= 0\end{aligned}$$

Coupled first order linear
differential equations

Solution I - Matrix method :

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

CF : Try

$$\begin{pmatrix} x \\ y \end{pmatrix} = \text{Re} \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$$

X, Y (complex) constants

$$\begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A.\Psi = 0$$

$$\Rightarrow \text{Det}[A] = 0$$

Eigenvalue equation

$$\begin{vmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{vmatrix} = 0$$

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \quad \Rightarrow \quad \left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm \left(\frac{k}{m}\right)$$

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm \left(\frac{k}{m}\right)$$

Eigenvalue equation

$$\omega_1^2 = \frac{g}{l} \quad \text{or} \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

Eigenvalues

$$\begin{pmatrix} x \\ y \end{pmatrix} = \text{Re} \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t} \Rightarrow \begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equation

$$\underline{\omega = \omega_1} : \begin{pmatrix} +\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & +\frac{k}{m} \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow X_1 = Y_1 = A_1 e^{i\phi_1}$$

Eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1)$$

1st Normal mode

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm \left(\frac{k}{m}\right)$$

Eigenvalue equation

$$\omega_1^2 = \frac{g}{l} \quad \text{or} \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

Eigenvalues

$$\begin{pmatrix} x \\ y \end{pmatrix} = \text{Re} \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t} \Rightarrow \begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equation

$$\underline{\omega = \omega_2} \quad \begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow X_2 = -Y_2 = A_2 e^{i\phi_2}$$

Eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

2nd Normal mode

$$m\ddot{x} = -mg\frac{x}{l} + k(y - x)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y - x)$$

General solution given by a superposition of the two independent (normal mode)solutions:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

Normal Modes

$$x + y = 2A_1 \cos(\omega_1 t + \phi_1)$$

$$x - y = 2A_2 \cos(\omega_2 t + \phi_2)$$

}

N 1D linear differential
Equations...N normal modes

