Normal Modes, Wave Motion and the Wave Equation

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This course of twelve lectures covers material for the paper CP4: Differential Equations, Waves and Optics of the Preliminary Examination in Physics, and Moderations in Physics and Philosophy. Textbooks covering aspects of this course include "Waves" by C.A. Coulson and A. Jeffrey, Longman, and "Vibrations and Waves" by A.P. French, MIT Introductory Physics Series. Many thanks to P.G. Irwin and G.G. Ross for providing their notes for this course.

Part 1 NORMAL MODES

1 Introduction

Many physical systems require more than one variable to quantify their configuration; for example a circuit may have two connected current loops, so one needs to know what current is flowing in each loop at each moment. Another example is a set of N coupled pendula each of which is a one-dimensional (1-D) oscillator. A set of differential equations– one for each variable – will determine the dynamics of such a system.

For a system of N coupled 1-D oscillators there exist N "normal modes" in which all oscillators move with the same frequency and thus have fixed amplitude ratios (if each oscillator is allowed to move in α-D, then αN normal modes exist). The normal mode is for whole system. Even though uncoupled angular frequencies of the oscillators are not the same, the effect of coupling is that all bodies can move with the same frequency. If the initial state of the system corresponds to motion in a normal mode then the oscillations continue in the normal mode. However in general the motion is described by a linear combination of all the normal modes; since the differential equations are linear such a linear combination is also a solution to the coupled linear equations.

The existence and nature of normal modes is best illustrated by some examples so we first turn to the solution of coupled linear equations.

2 Solution of coupled linear differential equations with constant coefficients.

Consider a set of differential equations that are linear and have constant coefficients. The procedure for solving them is a minor extension of the procedure for solving a single linear differential equation with constant coefficients. The steps are:

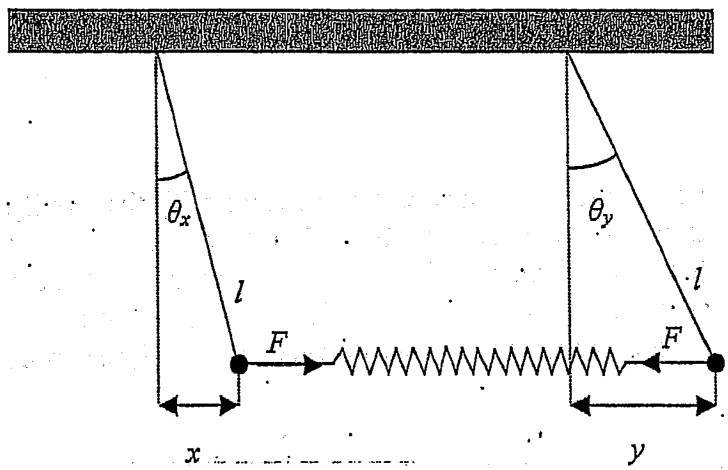
1. Arrange the equations so that terms on the left are all proportional to an unknown variable, and already known terms are on the right.

2. Find the general solution of the equations that are obtained by setting the right sides to zero. The result of this operation is the Complementary function (CF). For oscillatory solutions the CF is found by replacing the unknown variables by multiples of $e^{i\alpha t}$ (if t is the independent variable) and solving the resulting algebraic equations.

3. Find a particular integral by putting in a trial solution for each term – polynomial, exponential, etc. – on the right hand side.

3 Coupled Pendula

The first example of coupled linear differential equations is provided by two coupled pendula. Consider two massless rods of length l, which have bobs of mass m attached to the end, which are themselves connected by a spring.



Assumptions:

1) Assume that spring obeys Hooke's law and thus that the restoring force varies linearly with extension, i.e. $F = k(y-x)$
2) Assume the displacements from equilibrium positions are small such that the restoring force due to gravity for each pendulum is approximately given by $mg\tan\theta = mg\frac{x}{l}$ and acts along the line of masses. The equations of motion are then:

$$m\ddot{x} = -mgx/l + k(y-x)$$
$$m\ddot{y} = -mgy/l - k(y-x) \qquad (3.1)$$

3.1 Matrix method of solution

We start with the general method of solution that applies to all coupled linear differential equations. As we will discuss there may be more direct methods in special cases. We first implement step 1 to write the equations of motion for the coupled pendula in a standard form

$$\begin{aligned} -\ddot{x}+\left(-\frac{g}{l}-\frac{k}{m}\right)x+\frac{k}{m}y=0 \\ \frac{k}{m}x-\ddot{y}+\left(-\frac{g}{l}-\frac{k}{m}\right)y=0 \end{aligned} \tag{3.2}$$

where the unknown variables are to the left. In this case there are no driving terms so the right hand side is zero. These equations may be written as a matrix equation

$$\begin{pmatrix} -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.3}$$

or

$$\mathbf{Ax}=\mathbf{0} \tag{3.4}$$

where $\mathbf{x}$ is the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A}$ the square symmetric matrix in Eq.(3.3). Since the RHS is zero we are only interested in finding the CF. We look for normal mode solutions where all elements oscillate with the same frequency. Particularly for cases in which both first and second order derivatives are present (as is the case for damped oscillators discussed below) it is best to solve the associated complex equation. Writing

$$\mathbf{x}=\mathrm{Re}\left(\mathbf{Y}\equiv\mathbf{X}e^{i\omega t}\right) \tag{3.5}$$

where

$$\mathbf{X}=\begin{pmatrix} X \\ Y \end{pmatrix} \tag{3.6}$$

and substituting into Eq. 13 we find, since the differential operators are real, the associated complex equation is given by

$$\mathbf{AY}=0 \tag{3.7}$$

where

$$\mathbf{A}=\begin{pmatrix} \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix} \tag{3.8}$$

or, equivalently, dividing by the factor $e^{i\omega t}$

$$\mathbf{AX}=0 \tag{3.9}$$

The solutions of Eq.(3.9) are either $\mathbf{X}=0$, which is not very interesting, or the determinant of the matrix $\mathbf{A}$ must be equal to zero. Hence

$$\left|\begin{pmatrix} \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\right| = 0 \tag{3.10}$$

~~leading to the "eigenvalue equation":~~

$$\omega^2-\left(\frac{g}{l}+\frac{k}{m}\right)=\pm\frac{k}{m} \tag{3.11}$$

From this we see that there are two normal mode frequencies, $\omega_{1,2}$, corresponding to the two independent solutions of the coupled differential equations, given by

$$\begin{aligned} \omega_1 &= \sqrt{g/l} \\ \omega_2 &= \sqrt{g/l+2k/m} \end{aligned} \tag{3.12}$$

(the $\pm$ ambiguity associated with the square root gives rise to the same sinusoidal solutions and so is ignored here). To complete the solution we need to find the normal mode amplitudes. These are found by solving Eq.(3.9) for $\mathbf{X}$, substituting each of the normal mode frequencies in turn. For $\omega=\omega_1$ we have

$$\begin{pmatrix} +\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & +\frac{k}{m} \end{pmatrix}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.13}$$

This determines the ratio of X_1 to Y_1 giving

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}=A_1e^{i\phi_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.14}$$

with A_1, ϕ_1 the real amplitude and phase. Similarly for $\omega=\omega_2$

$$\begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.15}$$

giving

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}=A_2e^{i\phi_2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.16}$$

Finally, since $\mathbf{x}=\mathrm{Re}\left(\mathbf{X}e^{i\omega t}\right)$ we have the two "normal mode" solutions

$$\mathbf{x}_{1,2}=\begin{pmatrix} 1 \\ \pm1 \end{pmatrix}A_{1,2}\cos(\omega_{1,2}t+\phi_{1,2}) \tag{3.17}$$

and hence, since the differential equations are linear, we can use the principle of superposition to write the general solution as a linear combination of the two normal mode solutions

$$\mathbf{x}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}A_1\cos\left(\omega_1t+\phi_1\right)+\begin{pmatrix} 1 \\ -1 \end{pmatrix}A_2\cos\left(\omega_2t+\phi_2\right) \tag{3.18}$$

The advantage of the matrix method is its general applicability, and the ease with which it may be applied to systems with more than two normal modes. The advantage of using the complex exponential is only evident if there is a mixture of single and double derivatives as in the case of a damped pendulum discussed below. In the undamped case just discussed it would be equally simple to start with a normal mode trial solution proportional to $\cos(\omega t+\phi)$.

3.2 Alternative methods of solution: Normal coordinates or Decoupling

The equations of motion for the coupled pendula are given by Eq.(3.1), rewritten here for convenience

$$\begin{aligned} m\ddot{x} &= -mgx/l + k(y-x) \\ m\ddot{y} &= -mgy/l - k(y-x) \end{aligned} \tag{3.19}$$

For simple coupled oscillator systems it is often possible to find the normal modes directly by taking obvious linear combinations of the equations of motion to obtain decoupled differential equations. These may then be independently solved for a linear combination of the position variables, in this case x and y. If this can be done it considerable simplifies the solution. The coupled pendula just discussed provides a simple example of this. If we add Eqs.(3.19) we find:

$$m(\ddot{x}+\ddot{y}) = -\frac{mg}{l}(x+y) \tag{3.20}$$

or

$$\ddot{q}_1 = -\frac{g}{l}q_1 \tag{3.21}$$

where q_1 is a *normal coordinate* here equal to $q_1 = (x+y)/\sqrt{2}$ (The normalisation factor $1/\sqrt{2}$ is chosen to give a standard form for the kinetic energy when expressed in terms of the normal modes – see Eq.(3.41)).

Eq.(3.21) describes simple harmonic motion which may be trivially solved to give:

$$q_1 = \sqrt{2}A_1\cos(\omega_1 t+\phi_1) \tag{3.22}$$

where $\omega_1 = \sqrt{g/l}$ is the first normal frequency found earlier and we have chosen the integration constants to agree with those found using the matrix method.
Similarly, if we subtract Eqs.(3.19) we find:

$$m(\ddot{x}-\ddot{y}) = -\frac{mg}{l}(x-y) - 2k(x-y) \tag{3.23}$$

or

$$\ddot{q}_2 = -\left(\frac{g}{l}+\frac{2k}{m}\right)q_2 \tag{3.24}$$

where q_2 is another normal coordinate, equal to $q_1 = (x-y)/\sqrt{2}$. Eq.(3.24) also describes simple harmonic motion and thus

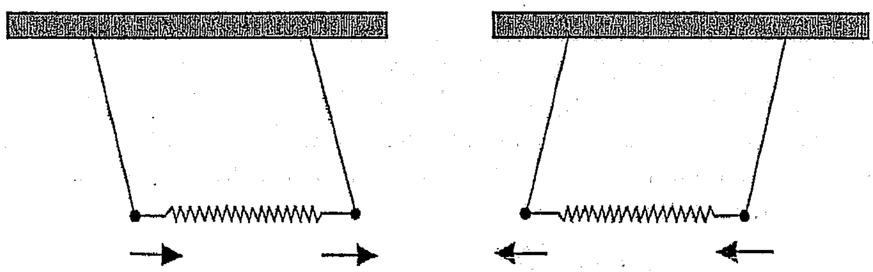
$$q_2 = \sqrt{2}A_2\cos(\omega_2 t+\phi_2) \tag{3.25}$$

where $\omega_2 = \sqrt{g/l + 2k/m}$ is the second normal frequency found earlier.

To extract the original position variables x and y we note that $x = (q_1 + q_2)/\sqrt{2}$ and $y = (q_1 - q_2)/\sqrt{2}$ and hence

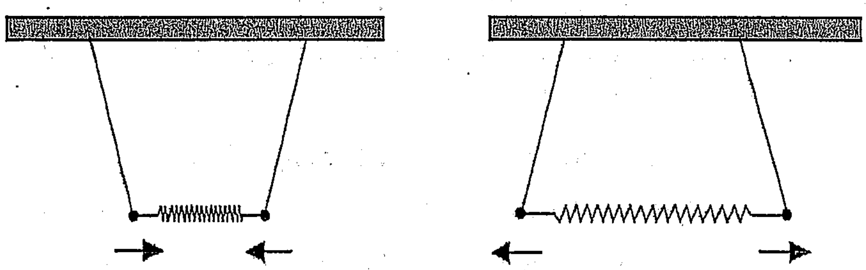
$$\begin{aligned} x(t) &= A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \\ y(t) &= A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \end{aligned} \tag{3.26}$$

which is identical to the general solutions Eq.(3.18) derived by the matrix method. From this is easy to identify the motion corresponding to the normal modes. For the case only the first normal mode is excited $A_2 = 0$ and the motion is shown in the figure below showing the two masses move together.



Normal mode 1

The second normal mode corresponds to the case $A_1 = 0$ and for it the masses move in opposite directions.



Normal Mode 2

This method of solution can lead to quick solutions for the normal frequencies if the suitable linear combination of parameters can be spotted. For simple cases like this it is easy but not for more complicated systems. This technique is also known as *decoupling*.

3.3 Initial conditions

The values of the integration constants A_i, ϕ_i are determined from the initial conditions of the system. As is shown in the following examples this can lead to a single normal mode being excited or to a combination of normal modes.

Example (a) – Normal Mode Excitation

Suppose that at t = 0, x = a, y = a and the masses are initially at rest. Equating the initial positions to Eqs.(3.26) implies:

$$\begin{aligned} x(0) &= A_1\cos\phi_1 + A_2\cos\phi_2 = a \\ y(0) &= A_1\cos\phi_1 - A_2\cos\phi_2 = a \end{aligned} \tag{3.27}$$

which implies $A_1\cos\phi_1 = a$, $A_2\cos\phi_2 = 0$. Equating the initial velocities to zero gives

$$\begin{aligned} \dot{x}(0) &= A_1\omega_1\sin\phi_1 + A_2\omega_2\sin\phi_2 = 0 \\ \dot{y}(0) &= A_1\omega_1\sin\phi_1 - A_2\omega_2\sin\phi_2 = 0 \end{aligned} \tag{3.28}$$

giving $A_1 = a, A_2 = 0, \phi_1 = 0$. Hence the solution for $t > 0$ is

$$\begin{aligned} x &= a\cos\omega_1 t \\ y &= a\cos\omega_1 t \end{aligned} \tag{3.29}$$

and thus, c.f. Eq.(3.26), we see that only the first normal mode is excited, which is to be expected given the initial displacements. In addition, once in this normal mode, the system will remain in it indefinitely.

Example (b) – Normal Mode Excitation

Suppose that at t = 0, x = y = 0, and the masses are given initial velocities $\dot{x} = -\mathrm{v}, \dot{y} = \mathrm{v}$. This implies

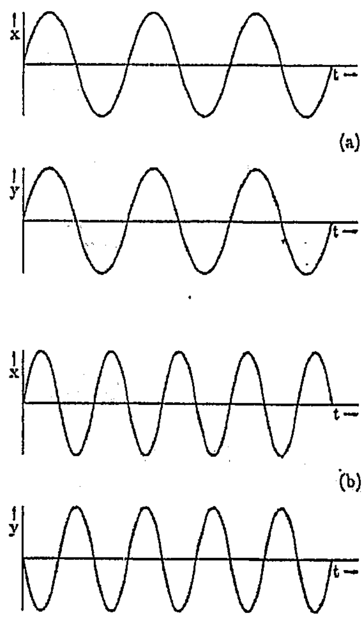
$$\begin{aligned} x(0) &= A_1\cos\phi_1 + A_2\cos\phi_2 = 0 \\ y(0) &= A_1\cos\phi_1 - A_2\cos\phi_2 = 0 \\ \dot{x}(0) &= A_1\omega_1\sin\phi_1 + A_2\omega_2\sin\phi_2 = -\mathrm{v} \\ \dot{y}(0) &= A_1\omega_1\sin\phi_1 - A_2\omega_2\sin\phi_2 = \mathrm{v} \end{aligned} \tag{3.30}$$

giving $A_1 = 0, A_2 = -\dfrac{v}{\omega_2}, \phi_2 = 0$ and thus the subsequent motion is:

$$\begin{aligned} x &= -\frac{v}{\omega_2}\sin\omega_2 t \\ y &= \frac{v}{\omega_2}\sin\omega_2 t \end{aligned} \tag{3.31}$$

Thus, c.f. Eq.(3.26), we see that these initial conditions excite the second normal mode only, in which the system will remain.

The motion in these two normal modes may also be summarised by the following figure:



where here the coupling is such that the frequency of the 2nd mode is higher than that of the first.

Example c – Non-Normal Behaviour – Beats

Suppose that at $t = 0$, $x = a$, $y = 0$ and the masses are initially at rest. This requires

$$\begin{aligned} x(0) &= A_1\cos\phi_1 + A_2\cos\phi_2 = a \\ y(0) &= A_1\cos\phi_1 - A_2\cos\phi_2 = 0 \\ \dot{x}(0) &= A_1\omega_1\sin\phi_1 + A_2\omega_2\sin\phi_2 = 0 \\ \dot{y}(0) &= A_1\omega_1\sin\phi_1 - A_2\omega_2\sin\phi_2 = 0 \end{aligned} \tag{3.32}$$

giving $A_1 = A_2 = \frac{a}{2}$, $\phi_1 = \phi_2 = 0$. Hence the solution for $t > 0$ is

$$\begin{aligned} x &= \frac{a}{2}\cos\omega_1 t + \frac{a}{2}\cos\omega_2 t \\ y &= \frac{a}{2}\cos\omega_1 t - \frac{a}{2}\cos\omega_2 t \end{aligned} \tag{3.33}$$

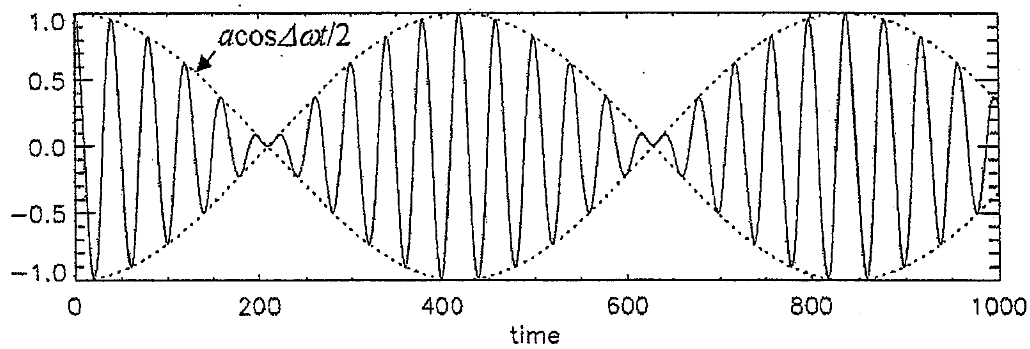
and thus both normal modes are excited. The solution for both x and y is then determined by the beating of the two terms with normal frequencies ω_1 and ω_2.

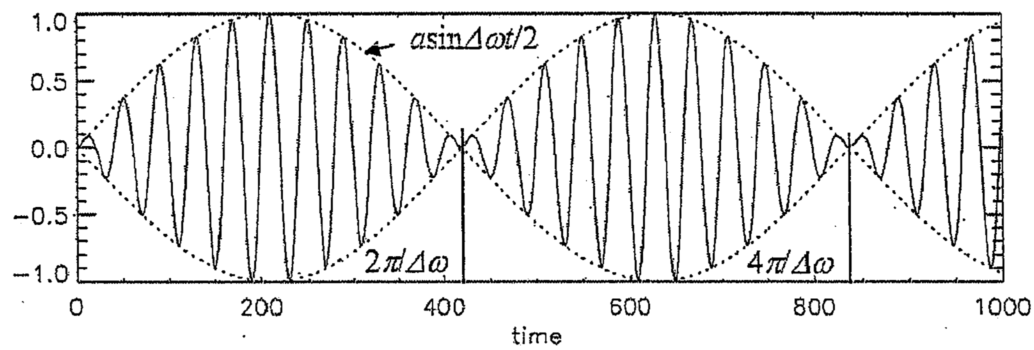
3.4 Beats

Eqs.(3.33) can be re-written using standard trigonometrical identities as:

$$x = a\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right) = a\cos(\bar{\omega}\,t)\cos(\Delta\omega\,t/2)$$
$$y = a\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right) = a\sin(\bar{\omega}\,t)\sin(\Delta\omega\,t/2) \qquad (3.34)$$

where $\bar{\omega} = \dfrac{\omega_1+\omega_2}{2}$ and $\Delta\omega = \omega_1 - \omega_2$. The form of x and y is shown in the figures below for the case $\Delta\omega \ll \bar{\omega}$.





versa, corresponding to a transfer of energy between the two pendula. Note also that one complete period of the envelope equals <u>two</u> beats.

3.5 Energy of Motion

Decoupling, to express the result in terms of normal modes, is also instructive when the energy of the system is considered. Consider first the potential energy, $V(x,y)$, of the coupled oscillators. Consider the forces acting on particle 1 which, c.f. Eq.(3.19), are given by $-mgx/l + k(y-x)$. This force may be written in terms of a partial derivative with respect to x of a potential $V(x,y)$:

$$-mgx/l + k(y-x) = -\frac{\partial V}{\partial x} \qquad (3.35)$$

Integrating this we find:

$$V = \frac{mg}{2l}x^2 + \frac{1}{2}kx^2 - kxy + f(y) \qquad (3.36)$$

where f is an unknown function of y.

The force, $-mgy/l-k(y-x)$, acting on particle 2 may be similarly be obtained from the potential energy giving

$$-mgy/l-k(y-x)=-\frac{\partial V}{\partial y}=kx-\frac{df}{dy} \tag{3.37}$$

where we have used Eq.(3.36) to compute the partial derivative. Integrating this equation determines $f(y)$ and inserted in Eq.(3.36) gives the total kinetic energy of the system (up to an undetermined constant)

$$V=\frac{1}{2}m\left(\frac{g}{l}+\frac{k}{m}\right)\left(x^2+y^2\right)-kxy \tag{3.38}$$

Consider now the kinetic energy K. This is given simply by:

$$K=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \tag{3.39}$$

While the expression for K is straightforward, that for V is rather more complex. However if we substitute for the normal coordinates: $x=\left(q_1+q_2\right)/\sqrt{2}$ and $y=\left(q_1-q_2\right)/\sqrt{2}$, then the V may be re-expressed as:

$$\begin{aligned}V&=\frac{1}{2}m\frac{g}{l}q_1^2+\frac{1}{2}m\left(\frac{g}{l}+\frac{2k}{m}\right)q_2^2\\&=\frac{1}{2}m\omega_1^2q_1^2+\frac{1}{2}m\omega_2^2q_2^2\end{aligned} \tag{3.40}$$

where ω_1 and ω_2 are the normal mode angular frequencies. Similarly K may be rewritten as:

$$K=\frac{1}{2}m\left(\dot{q}_1^2+\dot{q}_2^2\right) \tag{3.41}$$

and the total energy is then:

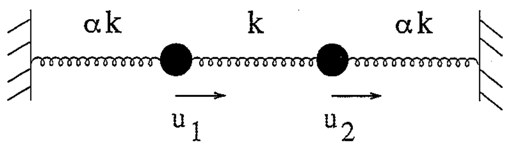
$$\begin{aligned}E&=\frac{1}{2}m\omega_1^2q_1^2+\frac{1}{2}m\omega_2^2q_2^2+\frac{1}{2}m\dot{q}_1^2+\frac{1}{2}m\dot{q}_2^2\\&=\left(\frac{1}{2}m\omega_1^2q_1^2+\frac{1}{2}m\dot{q}_1^2\right)+\left(\frac{1}{2}m\omega_2^2q_2^2+\frac{1}{2}m\dot{q}_2^2\right)\\&=E_1+E_2=\sum_{1}^{N=2}E_n\end{aligned} \tag{3.42}$$

One may see from Eqs.(3.40),(3.41) and (3.42) that the energies separate into the individual energies of two decoupled simple harmonic oscillators corresponding to the motion of the two normal modes. The is an example of Parseval's theorem which states that **the total energy of the system is the sum of the energies of the normal modes.**

4 Further examples of coupled oscillators

4.1 Spring-mass systems

As a next example of calculation of normal modes in mechanical systems with multiple degrees of freedom, we consider the system pictured below, consisting of two identical masses m constrained to move along a straight line on a smooth table under the action of three springs:



The middle spring connecting the two masses has elastic constant k; the end springs are fastened to fixed supports and have elastic constant αk, where α is a real number. Let the displacements of the two masses from equilibrium be u_1 and u_2.

Let us now a) write down the equations of motion of the system; b) determine the normal frequencies and normal modes of the system; c) express the energy of the system in terms of normal modes; d) solve motion for initial conditions $u_1 = u_0$, $u_2 = 0$, $\dot{u}_1 = 0$, $\dot{u}_2 = 0$.

The equations of motion are given by the two coupled 2nd-order linear differential equations

$$m\ddot{u}_1 = -\alpha k u_1 - k(u_1 - u_2) \ ,$$
$$m\ddot{u}_2 = -\alpha k u_2 + k(u_1 - u_2) \ .$$

If we now sum and subtract the two equations, and introduce new coordinates

$$S = \frac{u_1 + u_2}{\sqrt{2}} \ , \tag{1}$$

$$D = \frac{u_1 - u_2}{\sqrt{2}} \ , \tag{2}$$

we get

$$m\ddot{S} + \alpha k S = 0 \ ,$$
$$m\ddot{D} + (\alpha + 2)kD = 0 \ .$$

These are two decoupled equations of simple harmonic oscillators with frequencies

$$\omega_S = \sqrt{\frac{\alpha k}{m}} \ , \tag{3}$$

$$\omega_D = \sqrt{\frac{(\alpha + 2)k}{m}} \ . \tag{4}$$

Eqs. (3),(4) give the normal frequencies of the system, and the normal coordinates in Eqs. (1),(2) identify its normal modes.

The general solution of the equations of motion is given by

$$S(t) = C_S \sin(\omega_S t + \varphi_S) \ ,$$
$$D(t) = C_D \sin(\omega_D t + \varphi_D) \ ,$$

that is,

$$u_1(t) = \frac{1}{\sqrt{2}} \ [C_S \sin(\omega_S t + \varphi_S) + C_D \sin(\omega_D t + \varphi_D)] \ ,$$
$$u_2(t) = \frac{1}{\sqrt{2}} \ [C_S \sin(\omega_S t + \varphi_S) - C_D \sin(\omega_D t + \varphi_D)] \ .$$

The kinetic energy and the potential energy of the system are given respectively by

$$\begin{aligned} K &= \frac{1}{2} m \ (\dot{u}_1^2 + \dot{u}_2^2) \\ &= \frac{1}{2} m \ (\dot{S}^2 + \dot{D}^2) \ , \end{aligned}$$

$$\begin{aligned} V &= \frac{1}{2} \alpha k u_1^2 + \frac{1}{2} k \ (u_2 - u_1)^2 + \frac{1}{2} \alpha k u_2^2 \\ &= \frac{1}{2} \alpha k S^2 + \frac{1}{2} (\alpha + 2) k D^2 \\ &= \frac{1}{2} m \omega_S^2 S^2 + \frac{1}{2} m \omega_D^2 D^2 \ . \end{aligned}$$

The total energy is given by

$$\begin{aligned} E &= \underbrace{\frac{1}{2} m\dot{S}^2 + \frac{1}{2} m\omega_S^2 S^2}_{E_1} + \underbrace{\frac{1}{2} m\dot{D}^2 + \frac{1}{2} m\omega_D^2 D^2}_{E_2} \\ &= E_1 + E_2 \ , \end{aligned}$$

that is, the sum of the energies of each normal mode.

Given the initial conditions at time $t = 0$

$$u_1 = u_0 \ , \quad u_2 = 0 \ , \quad \dot{u}_1 = 0 \ , \quad \dot{u}_2 = 0 \ ,$$

the solution satisfying these conditions is given by

$$u_1(t) = \frac{1}{2} \ u_0 \, (\cos \omega_S t + \cos \omega_D t) \ ,$$

$$u_2(t) = \frac{1}{2} \ u_0 \, (\cos \omega_S t - \cos \omega_D t) \ .$$

Example

Three equal masses m are constrained to move on a circle without friction, subject to the action of three springs of elastic constant k connecting the three masses pairwise to each other. The equations of motion of the system are

$$m\ddot{u}_1 = -k(u_1 - u_2) + k(u_3 - u_1) \ ,$$
$$m\ddot{u}_2 = -k(u_2 - u_3) + k(u_1 - u_2) \ ,$$
$$m\ddot{u}_3 = -k(u_3 - u_1) + k(u_2 - u_3) \ .$$

Determine normal frequencies and normal coordinates of the system.
Verify that the answer is given by

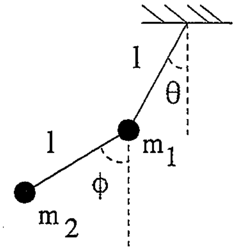
$$\omega_1 = 0 \ ,$$
$$\omega_2 = \omega_3 = \sqrt{\frac{3k}{m}} \ ,$$

and

$$q_1 = \frac{1}{\sqrt{3}} \, (u_1 + u_2 + u_3) \ ,$$
$$q_2 = \frac{1}{\sqrt{6}} \, (u_1 - 2u_2 + u_3) \ ,$$
$$q_3 = \frac{1}{\sqrt{2}} \, (u_1 - u_2) \ .$$

4.2 Double pendulum

Consider the double pendulum in the figure below, consisting of a mass m_1 hanging by a light string of length l from a rigid support, and a second mass m_2 hanging by an identical string of length l from m_1. The pendulum is constrained to swing in a vertical plane. Let the angle between the upper string and the vertical be θ, and that between the lower string and the vertical be ϕ. We consider the motion of the system for small angles about the equilibrium position. Let us write down the equations of motion and determine the normal modes of the system.



For $\theta, \phi \ll 1$ we have

$$m_1 l^2 \ddot{\theta} = -m_1 g l \theta + m_2 g l (\phi - \theta) \ ,$$
$$m_2 l^2 (\ddot{\theta} + \ddot{\phi}) = -m_2 g l \phi \ .$$

Eliminating $\ddot{\theta}$ in the second equation by using the first equation, we get

$$m_1 l\ddot{\theta} = -(m_1+m_2)\, g\theta + m_2 g\phi \ ,$$
$$m_2 l\ddot{\phi} = -m_2 g\phi\, (1+m_2/m_1) + m_2 g\theta\, (1+m_2/m_1) \ .$$

We can rewrite this pair of equations in matrix form as

$$\frac{d^2}{dt^2}\begin{pmatrix}\theta\\ \phi\end{pmatrix} = -\frac{g}{l}\begin{pmatrix}(m_1+m_2)/m_1 & -m_2/m_1\\ -(m_1+m_2)/m_1 & (m_1+m_2)/m_1\end{pmatrix}\begin{pmatrix}\theta\\ \phi\end{pmatrix} \ . \tag{5}$$

To find the normal modes of the double pendulum, look for solutions

$$\begin{pmatrix}\theta\\ \phi\end{pmatrix} = \begin{pmatrix}\theta_0\\ \phi_0\end{pmatrix} e^{i\omega t} \tag{6}$$

and substitute Eq. (6) into Eq. (5). We get

$$\begin{pmatrix}-\omega^2 + (g/l)(m_1+m_2)/m_1 & -(g/l)(m_2/m_1)\\ -(g/l)(m_1+m_2)/m_1 & -\omega^2 + (g/l)(m_1+m_2)/m_1\end{pmatrix}\begin{pmatrix}\theta_0\\ \phi_0\end{pmatrix} = 0 \ .$$

Then the normal frequencies are determined by

$$\det\begin{pmatrix}-\omega^2 + (g/l)(m_1+m_2)/m_1 & -(g/l)(m_2/m_1)\\ -(g/l)(m_1+m_2)/m_1 & -\omega^2 + (g/l)(m_1+m_2)/m_1\end{pmatrix} = 0 \,.$$

This gives

$$\omega^2 - (g/l)(m_1+m_2)/m_1 = \pm\, (g/l)\sqrt{m_2(m_1+m_2)}/m_1 \ ,$$

that is,

$$\omega_\pm = \sqrt{\frac{g}{l}\,\frac{1}{1 \mp \sqrt{m_2/(m_1+m_2)}}} \ . \tag{7}$$

For $m_1 = m_2$, $\omega_\pm = \sqrt{(2\pm\sqrt{2})g/l}$. For $m_1 \gg m_2$, $\omega_+ \simeq \omega_- \simeq \sqrt{g/l}$, as m_1 swings nearly undisturbed by m_2. For $m_1 \ll m_2$, $\omega_+ \to \infty$, and $\omega_- \to \sqrt{g/(2l)}$, as m_2 swings nearly as from a string of length $2l$.

5.0 Coupled driven linear differential equations

We will next consider examples in which there is a driving term forcing the motion, following the steps outlined in Sec. 2 when obtaining the solution. The example that follows appears in chapter 5 of the lecture notes on Complex Numbers and Ordinary Differential Equations. For completeness we reproduce it here.

Example 5.3

Solve

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= 2\sin t + 3\cos t + 5\mathrm{e}^{-t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - y &= 3\cos t - 5\sin t - \mathrm{e}^{-t} \end{aligned} \quad \text{given} \quad \begin{aligned} x(0) = 2; &\quad y(0) = -3 \\ \dot{x}(0) = 0; &\quad \dot{y}(0) = 4 \end{aligned}$$

To find CF

Set $x = X\mathrm{e}^{\alpha t}$, $y = Y\mathrm{e}^{\alpha t}$

$$\Rightarrow \quad \begin{vmatrix} (\alpha^2+2)X & + & \alpha Y \\ \alpha X & + & (\alpha^2-1)Y \end{vmatrix} = 0 \quad \Rightarrow \quad \alpha^4 = 2$$

$$\Rightarrow \quad \alpha^2 = \pm\sqrt{2} \quad \Rightarrow \quad \alpha = \pm\beta,\ \pm\mathrm{i}\beta \quad (\beta \equiv 2^{1/4})$$

and $Y/X = -(\alpha^2+2)/\alpha$ so the CF is

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} \beta \\ 2+\sqrt{2} \end{pmatrix} e^{-\beta t} + X_b \begin{pmatrix} -\beta \\ 2+\sqrt{2} \end{pmatrix} e^{\beta t} + X_c \begin{pmatrix} i\beta \\ 2-\sqrt{2} \end{pmatrix} e^{-i\beta t} + X_d \begin{pmatrix} -i\beta \\ 2-\sqrt{2} \end{pmatrix} e^{i\beta t}$$

Notice that the functions multiplying X_c and X_d are complex conjugates of one another. So if the solution is to be real X_d has to be the complex conjugate of X_c and these two complex coefficients contain only two real arbitrary constants between them. There are four arbitrary constants in the CF because we are solving second-order equations in two dependent variables.

To Find PI

Set $(x, y) = (X, Y)e^{-t} \quad \Rightarrow$

$$\begin{aligned} X - Y + 2X &= 5 \\ -X + Y - Y &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X &= 1 \\ Y &= -2 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

Have $2\sin t + 3\cos t = \Re e(\sqrt{13}e^{i(t+\phi)})$, where $\cos\phi = 3/\sqrt{13}$, $\sin\phi = -2/\sqrt{13}$.
Similarly $3\cos t - 5\sin t = \Re e(\sqrt{34}e^{i(t+\psi)})$, where $\cos\psi = 3/\sqrt{34}$, $\sin\psi = 5/\sqrt{34}$
Set $(x, y) = \Re e[(X, Y)e^{it}]$ and require

$$\begin{aligned} -X + iY + 2X = X + iY &= \sqrt{13}e^{i\phi} \\ iX - Y - Y = iX - 2Y &= \sqrt{34}e^{i\psi} \end{aligned} \quad \Rightarrow \quad \begin{aligned} -iY &= \sqrt{13}e^{i\phi} + i\sqrt{34}e^{i\psi} \\ iX &= 2i\sqrt{13}e^{i\phi} - \sqrt{34}e^{i\psi} \end{aligned}$$

so

$$\begin{aligned} x &= \Re e(2\sqrt{13}e^{i(t+\phi)} + i\sqrt{34}e^{i(t+\psi)}) \\ &= 2\sqrt{13}(\cos\phi\cos t - \sin\phi\sin t) - \sqrt{34}(\sin\psi\cos t + \cos\psi\sin t) \\ &= 2[3\cos t + 2\sin t] - 5\cos t - 3\sin t \\ &= \cos t + \sin t \end{aligned}$$

Similarly

$$\begin{aligned} y &= \Re e(\sqrt{13}ie^{i(t+\phi)} - \sqrt{34}e^{i(t+\psi)}) \\ &= \sqrt{13}(-\sin\phi\cos t - \cos\phi\sin t) - \sqrt{34}(\cos\psi\cos t - \sin\psi\sin t) \\ &= 2\cos t - 3\sin t - 3\cos t + 5\sin t \\ &= -\cos t + 2\sin t. \end{aligned}$$

For the initial-value problem

$$\mathrm{PI}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad ; \quad \dot{\mathrm{PI}}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$\mathrm{CF}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ; \quad \dot{\mathrm{CF}}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

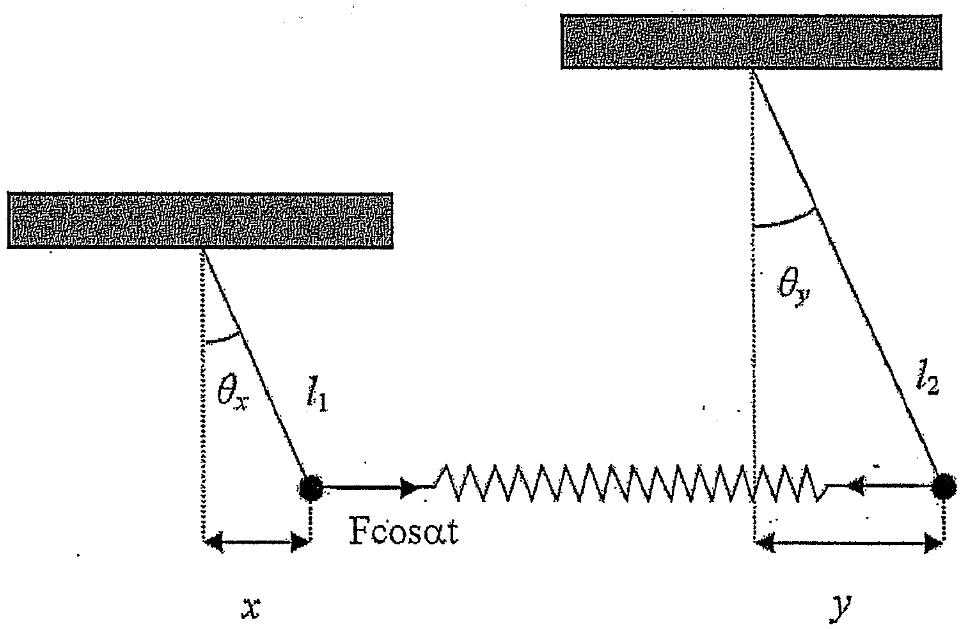
Therefore the solution satisfying the initial data is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \ .$$

5 Non-Identical Pendula, damped and with forced oscillations

Here we return to the coupled oscillator system to demonstrate how the matrix method provides a solution to the general case when the decoupling method is not straightforward to implement.

Consider the non-identical coupled oscillator system below with a force $F\cos\alpha t$ *acting on particle 1. Both* bobs are also subject to a frictional force equal to γ times their velocity.



The equations of motion are:

$$\begin{aligned} m_1\ddot{x} &= -\gamma\dot{x} - m_1 g x / l_1 + k(y-x) + F\cos\alpha t \\ m_2\ddot{y} &= -\gamma\dot{y} - m_2 g y / l_2 - k(y-x) \end{aligned} \tag{5.1}$$

This is equivalent to the matrix equation

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{\gamma}{m_1}\frac{d}{dt} + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2} + \frac{\gamma}{m_2}\frac{d}{dt} + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{Re}\left(e^{i\alpha t}\right) \tag{5.2}$$

The Complementary Function

The CF is found solving the equation with no driving term on the RHS. We look for a normal mode solution of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}\right) \tag{5.3}$$

Substituting this in the LHS of the matrix equation leads to the associated complex matrix equation

$$\begin{pmatrix} -\omega^2 + i\frac{\gamma}{m_1}\omega + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2 + i\frac{\gamma}{m_2}\omega + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{5.4}$$

The associated eigenvalue equation is

$$\begin{vmatrix} -\omega^2 + i\frac{\gamma}{m_1}\omega + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2 + i\frac{\gamma}{m_2}\omega + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{vmatrix} = 0 \tag{5.5}$$

Solving this equation will give the normal mode frequencies. Finally substituting these frequencies in turn in Eq.(5.4) determines the normal modes in the usual manner. For the case of arbitrary masses and pendula lengths this matrix method is the optimal one to find the normal frequencies as it is not possible simply to identify the normal co-ordinates and apply the decoupling method.

The Particular Integral

To find the particular integral we try

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\begin{pmatrix} P \\ Q \end{pmatrix} e^{i\alpha t}\right) \tag{5.6}$$

Substituting this in the matrix equation the associated complex equation is

$$\begin{pmatrix} -\alpha^2 + i\frac{\gamma}{m_1}\alpha + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2 + i\frac{\gamma}{m_2}\alpha + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \equiv \mathbf{MP} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.7}$$

where the factor $e^{i\alpha t}$ has been divided out of both sides. The solution to this equation is given by

$$\mathbf{P} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} = \mathbf{M}^{-1}\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.8}$$

where

$$\mathbf{M} = \begin{pmatrix} -\alpha^2 + i\frac{\gamma}{m_1}\alpha + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2 + i\frac{\gamma}{m_2}\alpha + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix} \tag{5.9}$$

Finally the PI is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\mathbf{M}^{-1} \frac{F}{m_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega t} \right) \tag{5.10}$$

This illustrates how the matrix method may be used to obtain the solution to the general driven coupled pendula system. However evaluating the solution is algebraically complicated so to illustrate the final steps we consider a relatively simple case.

5.1 The case $m_1 = m_2 = m, \quad l_1 = l_2 = l$ - Matrix method

CF

In this case the eigenvalue equation, Eq.(5.5), becomes

$$\begin{aligned} &\left(-\omega^2 + i\frac{\gamma}{m}\omega + \left(\frac{g}{l}\right)\right)\left(-\omega^2 + i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{2k}{m}\right)\right) \\ &\equiv \left(-\omega^2 + \omega_1^2 + i\frac{\gamma}{m}\omega \right)\left(-\omega^2 + \omega_2^2 + i\frac{\gamma}{m}\omega \right) = 0 \end{aligned} \tag{5.11}$$

with solutions

$$\overline{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2} \tag{5.12}$$

where $\omega_{1,2}$ are the normal frequencies for the case with no damping, c.f. Eq.(3.12).

For the case $\omega = \overline{\omega}_1$, corresponding to the first factor in Eq.(5.11) vanishing, $\left(-\overline{\omega}_1^2 + i\frac{\gamma}{m}\overline{\omega}_1 + \frac{g}{l} \right) = 0$ the eigenvector equation becomes

$$\begin{pmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{5.13}$$

implying

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A_1 e^{i\phi_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.14}$$

Similarly one readily finds the case $\omega = \overline{\omega}_2$ has its eigenvalues given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A_2 e^{i\phi_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5.15}$$

Putting this all together we have the complementary function

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \mathrm{Re}\left(A_1 e^{i\phi_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{\gamma}{2m}t} e^{i\sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2}} \right) + \mathrm{Re}\left(A_2 e^{i\phi_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{\gamma}{2m}t} e^{i\sqrt{\omega_2^2 - \left(\frac{\gamma}{2m}\right)^2}} \right) \\ &= e^{-\frac{\gamma}{2m}t}\left(A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2} + \phi_1 \right) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\sqrt{\omega_2^2 - \left(\frac{\gamma}{2m}\right)^2} + \phi_2 \right)\right) \end{aligned} \tag{5.16}$$

PI

For this choice of masses and lengths the matrix given in Eq.(5.9) is now

$$\mathbf{M}=\begin{pmatrix} -\alpha^2+i\frac{\gamma}{m}\alpha+\left(\frac{g}{l}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\alpha^2+i\frac{\gamma}{m}\alpha+\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix} \tag{5.17}$$

with inverse

$$\mathbf{M}^{-1}=\frac{1}{Det\,\mathbf{M}}\begin{pmatrix} -\alpha^2+i\frac{\gamma}{m}\alpha+\left(\frac{g}{l}+\frac{k}{m}\right) & +\frac{k}{m} \\ +\frac{k}{m} & -\alpha^2+i\frac{\gamma}{m}\alpha+\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix} \tag{5.18}$$

where

$$Det\,\mathbf{M}=\left(-\alpha^2+i\frac{\gamma}{m}\alpha+\left(\frac{g}{l}\right)\right)\left(-\alpha^2+i\frac{\gamma}{m}\alpha+\left(\frac{g}{l}+\frac{2k}{m}\right)\right) \tag{5.19}$$

It is convenient to rewrite this in the form

$$Det\,\mathbf{M}=B_1e^{-i\theta_1}.B_2e^{-i\theta_2} \tag{5.20}$$

where

$$B_{1,2}=\left(\left(-\alpha^2+\omega_{1,2}^2\right)^2+\left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}$$

$$\tan\theta_{1,2}=\frac{-\alpha\gamma/m}{\left(-\alpha^2+\omega_{1,2}^2\right)^2} \tag{5.21}$$

Then $\mathbf{M}^{-1}$ may be rewritten as

$$\mathbf{M}^{-1}=\frac{e^{i(\theta_1+\theta_2)}}{2B_1B_2}\begin{pmatrix} B_1e^{-i\theta_1}+B_2e^{-i\theta_2} & -B_1e^{-i\theta_1}+B_2e^{-i\theta_2} \\ -B_1e^{-i\theta_1}+B_2e^{-i\theta_2} & B_1e^{-i\theta_1}+B_2e^{-i\theta_2} \end{pmatrix} \tag{5.22}$$

so finally, using Eq.(5.10) we have

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix}&=\mathrm{Re}\left[\frac{F}{2m}\frac{e^{i(\theta_1+\theta_2)}}{B_1B_2}e^{i\alpha t}\begin{pmatrix} B_1e^{-i\theta_1}+B_2e^{-i\theta_2} \\ -B_1e^{-i\theta_1}+B_2e^{-i\theta_2} \end{pmatrix}\right] \\ &=\frac{F}{2mB_1B_2}\begin{pmatrix} B_1\cos(\alpha t+\theta_2)+B_2\cos(\alpha t+\theta_1) \\ -B_1\cos(\alpha t+\theta_2)+B_2\cos(\alpha t+\theta_1) \end{pmatrix}\end{aligned} \tag{5.23}$$

5.2 The case $m_1 = m_2 = m, \quad l_1 = l_2 = l$ - Decoupling method

For this choice of masses and lengths the decoupling method provides another way of identifying the normal modes and decoupling the differential equations for the driven oscillators. It is instructive to compare this to the matrix method. The coupled differential equations are

$$\begin{aligned} m_1\ddot{x} &= -\gamma\dot{x} - m_1 g x / l_1 + k(y-x) + F\cos\alpha t \\ m_2\ddot{y} &= -\gamma\dot{y} - m_2 g y / l_2 - k(y-x) \end{aligned} \tag{5.24}$$

The first normal mode

Adding the equations gives

$$(\ddot{x}+\ddot{y}) = -\frac{g}{l}(x+y) - \frac{\gamma}{m}(\dot{x}+\dot{y}) + \frac{F}{m}\cos\alpha t \tag{5.25}$$

or

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \frac{g}{l}q_1 = \frac{F}{\sqrt{2}m}\cos\alpha t = \frac{F}{\sqrt{2}m}\mathrm{Re}\, e^{i\alpha t} \tag{5.26}$$

CF

The auxiliary equation is

$$-\omega^2 + i\frac{\gamma}{m}\omega + \frac{g}{l} = 0 \tag{5.27}$$

with eigenvalues

$$\overline{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2} \tag{5.28}$$

as we found using the matrix method. The CF is then

$$q_1 = \sqrt{2}A_1 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2} + \delta_1\right) \tag{5.29}$$

where $q_1 = \frac{1}{\sqrt{2}}(x+y)$.

PI

To find the P.I. put $q_1 = \mathrm{Re}\left[C_1 \exp(i\alpha t)\right]$ then, c.f. Eq.(5.21):

$$\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l}\right)C_1 \equiv B_1 e^{-i\theta_1}C_1 = \frac{F}{\sqrt{2}m} \tag{5.30}$$

and thus

$$C_1 = \frac{F}{\sqrt{2}mB_1}\exp(i\theta_1) \tag{5.31}$$

Hence the PI for the normal coordinate q_1 is given by:

$$q_1 = \frac{F}{\sqrt{2}mB_1}\cos(\alpha t + \theta_1) \tag{5.32}$$

The second normal mode

Subtracting the equations of motion (Eq.(5.24)) gives:

$$(\ddot{x} - \ddot{y}) + \frac{\gamma}{m}(\dot{x} - \dot{y}) + \frac{g}{l}(x - y) + \frac{2k}{m}(x - y) = \frac{F}{m}\cos\alpha t \tag{5.33}$$

or

$$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \left(\frac{g}{l} + \frac{2k}{m}\right)q_2 = \frac{F}{\sqrt{2}m}\cos\alpha t = \mathrm{Re}\left[\frac{F}{\sqrt{2}m}\exp(i\alpha t)\right] \tag{5.34}$$

where $q_2 = \frac{1}{\sqrt{2}}(x - y)$

CF

In a similar manner we readily find that the complementary function is given by

$$q_2 = \sqrt{2}A_2 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_2^2 - \left(\frac{\gamma}{2m}\right)^2} + \delta_2\right) \tag{5.35}$$

PI

To find the P.I. put $q_2 = \mathrm{Re}\left[C_2\exp(i\alpha t)\right]$ then:

$$\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l} + \frac{2k}{m}\right)C_2 \equiv B_2 e^{i\theta_2}C_2 = \frac{F}{\sqrt{2}m} \tag{5.36}$$

and thus

$$C_2 = \frac{F}{\sqrt{2}mB_2}\exp(i\phi_2) \tag{5.37}$$

Hence the normal coordinate q_2 is given by:

$$q_2 = \frac{F}{\sqrt{2}mB_2}\cos(\alpha t + \phi_2) \tag{5.38}$$

It is easy to solve for x and y giving

$$\begin{aligned}
x &= A_1 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_1^2-\left(\frac{\gamma}{2m}\right)^2}+\delta_1\right)+A_2 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_2^2-\left(\frac{\gamma}{2m}\right)^2}+\delta_2\right)\\
&+\frac{F}{2mB_1}\cos(\alpha t+\theta_1)+\frac{F}{2mB_2}\cos(\alpha t+\phi_2)\\
y &= A_1 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_1^2-\left(\frac{\gamma}{2m}\right)^2}+\delta_1\right)-A_2 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_2^2-\left(\frac{\gamma}{2m}\right)^2}+\delta_2\right)\\
&+\frac{F}{2mB_1}\cos(\alpha t+\theta_1)-\frac{F}{2mB_2}\cos(\alpha t+\phi_2)
\end{aligned}\tag{5.39}$$

in agreement with the result obtained by the matrix method, Eqs.(5.16) and (5.23)

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5.3 The case $m_1 = m_2 = m$, $l_1 \neq l_2$, no damping, no driving force.

The final example we shall consider is the case that the masses are equal and there is no driving force but the pendula lengths differ. Fron Eq.(5.5) the eigenvalue equation is

$$\begin{vmatrix} -\omega^2+\left(\frac{g}{l_1}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+\left(\frac{g}{l_2}+\frac{k}{m}\right) \end{vmatrix}=0 \tag{5.40}$$

Putting:

$$\begin{aligned}
A &= g/l_1+k/m=\beta_1^2+k/m\\
B &= -k/m\\
C &= g/l_2+k/m=\beta_2^2+k/m
\end{aligned}\tag{5.41}$$

gives

$$\omega_{1,2}^2=\frac{1}{2}\left[\left(\beta_1^2+\beta_2^2\right)+2k/m\pm\sqrt{\left(\beta_1^2-\beta_2^2\right)^2+(2k/m)^2}\right] \tag{5.42}$$

Substituting $\omega_{1,2}$ in Eq.(5.4) determines the eigenvectors $\mathbf{X}=\begin{pmatrix}x_0\\ y_0\end{pmatrix}$ up to an overall constant:

$$\frac{x_0}{y_0}=-\frac{m}{2k}\left[\left(\beta_1^2-\beta_2^2\right)\pm\sqrt{\left(\beta_1^2-\beta_2^2\right)^2+(2k/m)^2}\right] \tag{5.43}$$

N.B. $(x_0/y_0)_1$ for mode 1 and $(x_0/y_0)_2$ for mode 2 are related by:

$$\left(\frac{y_0}{x_0}\right)_1=-1/\left(\frac{y_0}{x_0}\right)_2\equiv r \tag{5.44}$$

The full solution is then given by

$$\mathbf{x}(t)=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\r\end{pmatrix}D\cos(\omega_1 t+\delta_1)+\begin{pmatrix}-r\\1\end{pmatrix}G\cos(\omega_2 t+\delta_2) \tag{5.45}$$

Suppose at $t = 0$, $\mathbf{x}(0)=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a\\0\end{pmatrix}$ and $\dot{\mathbf{x}}=0$. Having zero initial velocities means that $\delta_1=\delta_2=0$. Hence

$$\mathbf{x}(0)=\begin{pmatrix}a\\0\end{pmatrix}=D\begin{pmatrix}1\\r\end{pmatrix}+G\begin{pmatrix}-r\\1\end{pmatrix} \tag{5.46}$$

which may be solved to give

$$\begin{aligned} x(t)&=a\left[\cos\omega_1 t+r^2\cos\omega_2 t\right]/\left(1+r^2\right)\\ y(t)&=ar\left[\cos\omega_1 t-\cos\omega_2 t\right]/\left(1+r^2\right) \end{aligned} \tag{5.47}$$

This is a little more difficult to simplify but it can be shown that

$$\begin{aligned} x(t)&=a\cos(\bar{\omega}t)\cos(\Delta\omega t/2)-a\left(\frac{1-r^2}{1+r^2}\right)\sin(\bar{\omega}t)\sin(\Delta\omega t/2)\\ y(t)&=2ar\sin(\bar{\omega}t)\sin(\Delta\omega t/2)/\left(1+r^2\right) \end{aligned} \tag{5.48}$$

From this one sees that $|x|$ varies between a and $\left(\frac{1-r^2}{1+r^2}\right)a$ and $|y|$ varies between 0 and $\left(\frac{2r}{1+r^2}\right)a$. Hence, unlike the case for equal length pendula, there is an incomplete transfer of energy. This is clear from the plot of

$x(t)$ and $y(t)$:

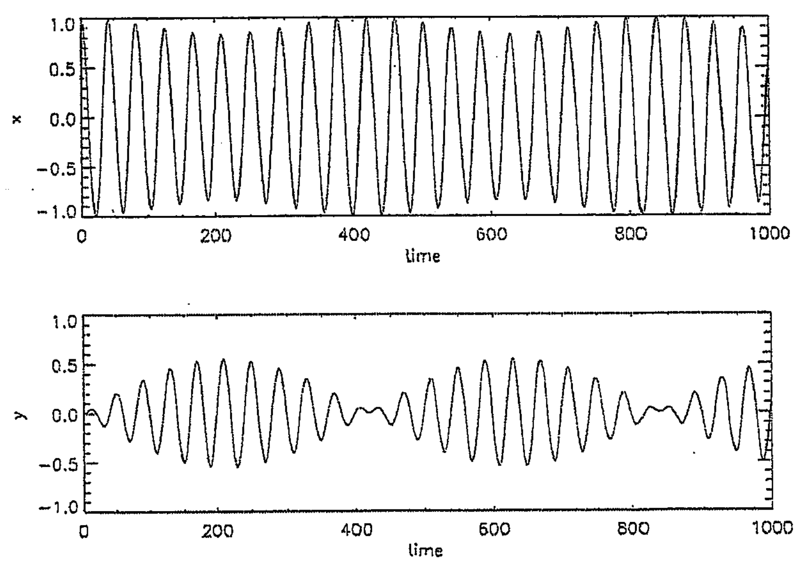


Figure showing beats of non-identical pendula. Note the incomplete energy transfer.

5.4 Diagrammatic Representation of Normal Modes

The normal mode motion is specified by the ratio x_0/y_0. We can represent this by a unit-length vector $\mathbf{v} = (x_0\mathbf{i} + y_0\mathbf{j}) / \sqrt{x_0^2 + y_0^2}$. For the case of two normal modes there are two vectors.

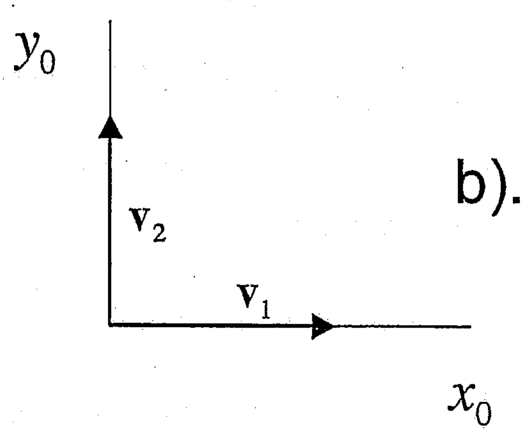
Consider case of non-identical pendula discussed in Section 5.3. In various limits these eigenvalues defining the normal modes are given by

		$\mathbf{v}_1$		$\mathbf{v}_2$
(a) For $k/m \to 0$	$x_0/y_0. \to$	$-\infty$	or	0
(b) For $k/m \to \infty$	$x_0/y_0. \to$	-1	or	1

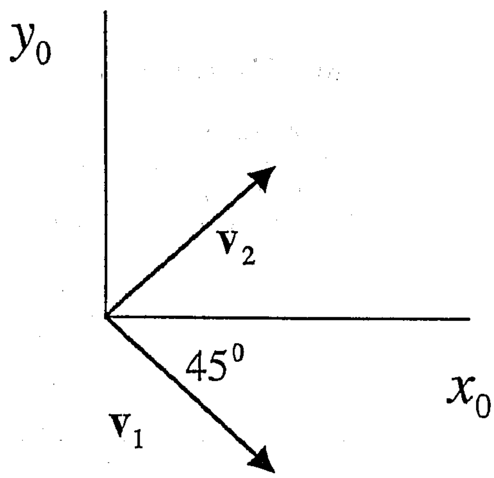
(c) Intermediate k/m $\quad \dfrac{y_0}{x_0} = \tan\theta = \dfrac{-2k/m}{(\beta_1^2 - \beta_2^2) \pm \sqrt{(\beta_1^2 - \beta_2^2)^2 + (2k/m)^2}}$

The corresponding graphical representation is given by

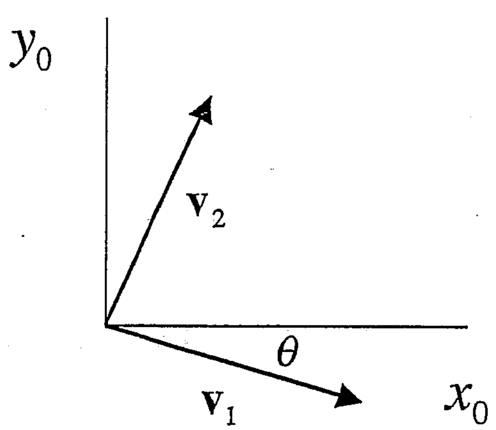
a).



b).



c).



Normal Modes, Wave Motion and the Wave Equation

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Part 2 WAVE MOTION AND THE WAVE EQUATION

6 Introduction

The answer to the question "Why should we learn about waves" is simple: Waves are everywhere! For example

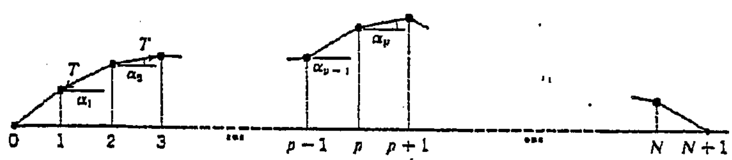
Strings	Violin
Membranes	Drum
Air	Sound
Optics	Interference and diffraction
E.M.	Radio, T.V., ...
Quantum mechanics	Uncertainty principle
	α-decay
Seismology	Earthquakes

In the physics course you will study many of these so it is important that you should have a good understanding of the mathematical description of waves.

To start we will consider a simple case of a wave propagating along a string stretched along the x-axis. The string can have transverse vibrations corresponding to a displacement in the y-direction given by $y(x, t)$ at position x and time t. It is instructive first to consider a case similar to those we have been studying in which a system of masses on a stretched elastic string undergo transverse vibrations.

7 N coupled oscillators

Consider the transverse oscillations of N particles of mass m spaced equally along a flexible, elastic, massless string, which is under tension T.



(reproduced from French, 1971).

Assume the particles are displaced by small distances y_i and thus the angles α_i are small too. In this case the length of the string between the particles is increased to $l' = l / \cos\alpha_i \approx l\left(1+\alpha_i^2/2\right)$ i.e. $l' \approx l$ and the tension in the string remains constant.

Consider the p^{th} particle above. The force acting in the y-direction is

$$F = -T\sin\alpha_{p-1} + T\sin\alpha_p \qquad (7.1)$$

which may be approximated by

$$F \approx -\frac{T}{l}\left(y_p - y_{p-1}\right) + \frac{T}{l}\left(y_{p+1} - y_p\right) \qquad (7.2)$$

Hence the equation of motion of the p^{th} particle is

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} + y_{p-1}\right) = 0 \qquad (7.3)$$

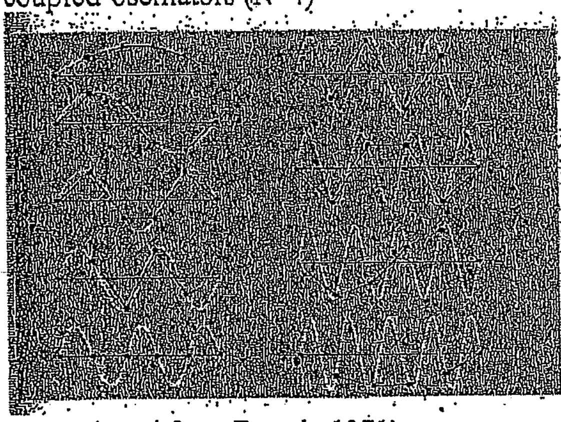
where $\omega_0^2 = T/ml$. We can write a similar equation for each of the N particles and thus we have N coupled differential equations and thus N normal modes. The solution is a linear combination of the normal modes and reads

$$y_p = \sum_{n=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right)\left(D_n \cos\omega_n t + E_n \sin\omega_n t\right) \qquad (7.4)$$

where

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \qquad (7.5)$$

It is instructive to consider what these normal modes look like. For example for four coupled oscillators (N=4)



(reproduced from French, 1971)

Clearly there 4 normal modes in all. Note that n = 6, 7, 8, 9 repeat patterns of n = 4, 3, 2,1 with opposite sign. This illustrates the start of the wave pattern, shown in the diagram in white, that we shall see occurs as $N \to \infty$ when the continuous distribution of masses becomes a string.

Derivation of Eqs. (7.4),(7.5)

Here we obtain the solution (7.4)-(7.5) to the set of N coupled equations (7.3)

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0 , \quad p = 1, \ldots, N$$

describing the motion of the N masses along the string. The ends of the string are held fixed, corresponding to $y_0 = y_{N+1} = 0$.
We look for solutions of the form

$$y_p(t) = \text{Re}[Y_p e^{i\omega t}] . \tag{1}$$

Substituting this into the equations of motion, we obtain

$$(-\omega^2 + 2\omega_0^2)Y_p - \omega_0^2(Y_{p+1} + Y_{p-1}) = 0 ,$$

that is,

$$\frac{Y_{p+1} + Y_{p-1}}{Y_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} . \tag{2}$$

Here the right hand side is constant for any fixed ω. The left hand side must be independent of p. Note that

$$Y_p = C \sin p\theta$$

satisfies this condition (for some suitable θ), because it gives

$$Y_{p+1} + Y_{p-1} = C[\sin(p+1)\theta + \sin(p-1)\theta] = 2C \sin p\theta \cos\theta ,$$

i.e.,

$$\frac{Y_{p+1} + Y_{p-1}}{Y_p} = 2\cos\theta . \tag{3}$$

The fixed-end condition $y_{N+1} = 0$ requires $\sin(N+1)\theta = 0$, that is,

$$\theta = n\pi/(N+1) \quad , \qquad n \text{ integer} . \tag{4}$$

Therefore,

$$Y_p = C \sin \frac{pn\pi}{N+1} . \tag{5}$$

By identifying Eq. (2) and Eq. (3), with θ given in Eq. (4), we get

$$2\cos\frac{n\pi}{N+1} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} .$$

Solving this for ω, we determine the normal frequencies ω_n:

$$\omega_n = 2\omega_0 \sin\frac{n\pi}{2(N+1)} ,$$

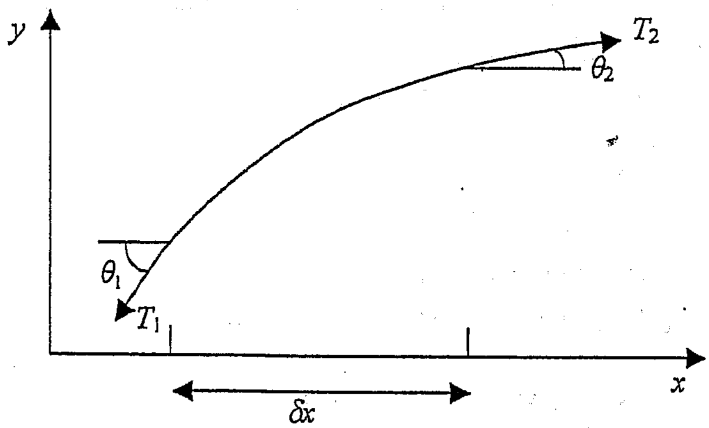
which is Eq. (7.5). Note that n here runs over $n = 1, \ldots, N$, because for $n = 0$ and $n = N+1$ all amplitudes (5) vanish, and for higher n we just reobtain the same frequencies, e.g., $\omega_{N+2} = \omega_N$, $\omega_{N+3} = \omega_{N-1}$, and so on. Thus there are N normal frequencies.
The general solution can be written as a linear combination of the normal modes. Using (1),(5), we have

$$y_p = \sum_{n=1}^{N} c_n \sin\frac{pn\pi}{N+1} \cos(\omega_n t + \phi_n) ,$$

which is the solution in Eq. (7.4).

8 The wave equation – Transverse waves on a string

The simplest example of wave motion is that of transverse displacements of an elastic string. Consider the diagram of small portion of string (θ_1 and θ_2 small). Suppose the string has linear density (kg/m) ρ.



For small angle of displacement and transverse oscillations, as we discussed above, the tension, T, is approximately constant along the string. Resolving the forces acting on the portion of string in the y-direction we have, from Newton's second law

$$T\sin\theta_2 - T\sin\theta_1 = (\rho\delta x)\frac{\partial^2 y}{\partial t^2} \tag{8.1}$$

For small angles $\sin\theta \approx \tan\theta = \frac{\partial y}{\partial x}$ and hence

$$T\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \rho\delta x\frac{\partial^2 y}{\partial t^2} \tag{8.2}$$

The final step is to replace $\frac{\partial y}{\partial x}$ by the leading terms in its Taylor series

$$\left(\frac{\partial y}{\partial x}\right)_2 = \left(\frac{\partial y}{\partial x}\right)_1 + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\delta x + \ldots \tag{8.3}$$

Using this Eq.(8.2) becomes

$$T\left(\frac{\partial^2 y}{\partial x^2}\right)\delta x = \rho\frac{\partial^2 y}{\partial t^2}\delta x \tag{8.4}$$

and so we obtain the *Wave Equation*

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \tag{8.5}$$

As we shall discuss this describes a wave moving with velocity $c = \sqrt{T/\rho}$ (hence larger tension or lighter string leads to faster waves).

9 D'Alembert's Solution

Following from Eq.(8.5) the wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{9.1}$$

This is most easily solved by changing variables to

$$\begin{aligned} u &= x - ct \\ \mathrm{v} &= x + ct \end{aligned} \tag{9.2}$$

The wave equation may then be written in terms of these new variables by application of the chain rule. i.e. since $y(x,t) = y(u,\mathrm{v})$

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial \mathrm{v}}\frac{\partial v}{\partial x} \\ &= \frac{\partial y}{\partial u} + \frac{\partial y}{\partial \mathrm{v}} \end{aligned} \tag{9.3}$$

similarly

$$\begin{aligned} \frac{\partial y}{\partial t} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial \mathrm{v}}\frac{\partial v}{\partial t} \\ &= -c\frac{\partial y}{\partial u} + c\frac{\partial y}{\partial \mathrm{v}} \end{aligned} \tag{9.4}$$

Differentiating again we find:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial \mathrm{v}} + \frac{\partial^2 y}{\partial \mathrm{v}^2} \tag{9.5}$$

and

$$\frac{\partial^2 y}{\partial t^2} = c^2\left[\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial \mathrm{v}} + \frac{\partial^2 y}{\partial \mathrm{v}^2}\right] \tag{9.6}$$

Hence substituting into the wave equation we find

$$\frac{\partial^2 y}{\partial u \partial \mathrm{v}} = 0 \tag{9.7}$$

from which we may deduce that

$$y(u,v) = f(u) + g(\mathrm{v}) \tag{9.8}$$

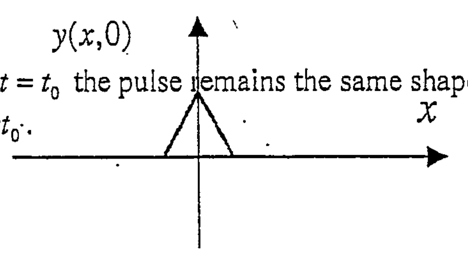
or

$$y(x,t) = f(x-ct) + g(x+ct) \tag{9.9}$$

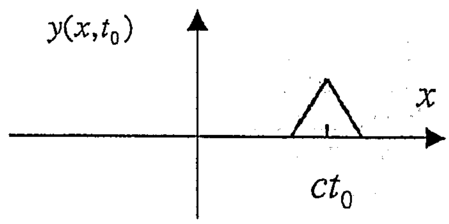
where f and g are *any* functions of u and v. This is the general solution to the wave equation. The functions f and g are determined by the initial conditions as we shall show in Section 9.3. However first let us consider the meaning of this solution.

9.1 Travelling waves

Let us illustrate the solution just obtained by choosing $y(x,t) = f(x-ct)$ at $t = 0$ to be a pulse centered at the origin.

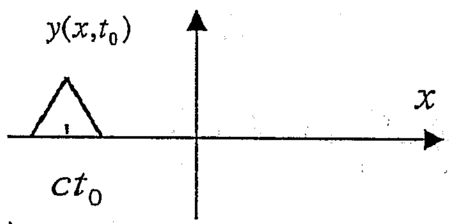


Then at a later time $t = t_0$ the pulse remains the same shape but is translated to the right by a distance ct_0.



Thus $y(x,t)=f(x-ct)$ represents "travelling wave" moving to the right.

Now consider the case $y(x,t)=g(x+ct)$ at $t=0$ to be a pulse centered at the origin as in the first diagram. Then at $t=t_0$ the pulse remains the same shape but is translated to the *left* by a distance ct_0.



Thus $y(x,t)=g(x+ct)$ represents "travelling wave" moving to the left.

9.2 D'Alembert's solution with boundary conditions

As we have just seen the functions f and g are determined by the initial conditions of the wave. These can be incorporated in d'Alembert's solution in a straightforward way. Suppose at time $t = 0$, the wave has an initial displacement $U(x)$ and an initial velocity $V(x)$

$$y(x,0)=f(x)+g(x)=U(x) \tag{9.10}$$

$$\frac{\partial y(x,0)}{\partial t}=-cf'(x)+cg'(x)=V(x) \tag{9.11}$$

integrating Eq. (9.11) gives:

$$f(x)-g(x)=-\frac{1}{c}\int_b^x V(x)\,dx \tag{9.12}$$

Adding Eqs (9.10) and (9.12) leads to

$$f(x)=\frac{1}{2}U(x)-\frac{1}{2c}\int_b^x V(x)\,dx \tag{9.13}$$

Subtracting Eqs. Eqs (9.10) and (9.12) leads to

$$g(x)=\frac{1}{2}U(x)+\frac{1}{2c}\int_{b}^{x}V(x)\,dx \qquad (9.14)$$

Hence combining these to form $y(x,t)$ we find:

$$y(x,t)=\frac{1}{2}\left[U(x-ct)+U(x+ct)\right]+\frac{1}{2c}\left[\int_{b}^{x+ct}V(x)\,dx-\int_{b}^{x-ct}V(x)\,dx\right] \qquad (9.15)$$

or

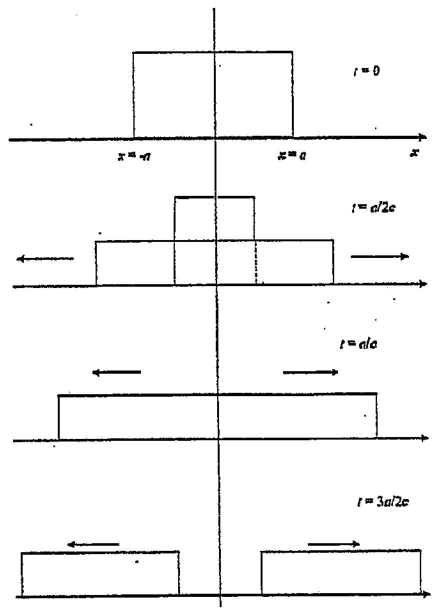
$$y(x,t)=\frac{1}{2}\left[U(x-ct)+U(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}V(x)\,dx \qquad (9.16)$$

9.3 An example of D'Alembert's solution

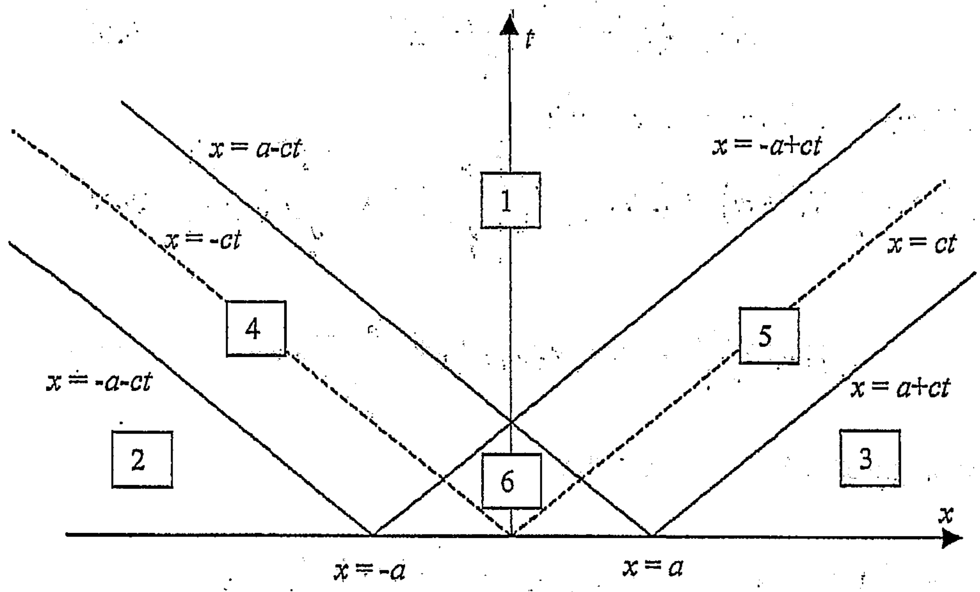
A stretched string is released from rest (*i.e.* $V(x) = 0$) with an initial square displacement. Hence from Eq. (9.16)

$$y(x,t)=\frac{1}{2}\left[U(x-ct)+U(x+ct)\right] \qquad (9.17)$$

The resulting evolution is shown in the Figure below



This figure can also be represented on a space-time (x,t) domain. Let $y(x,t)$ point out of the paper.



In regions 1, 2, and 3, $y(x,t) = 0$ for all x, t

In region 5, $y(x,t)=\frac{1}{2}u(x-ct)$ $\quad -a \le x-ct \le a$

In region 4, $y(x,t)=\frac{1}{2}u(x+ct)$ $\quad -a \le x+ct \le a$

In region 6, $y(x,t)=\frac{1}{2}\left[u(x-ct)+u(x+ct)\right]$ $\quad x-ct > -a$

$x+ct < a$

10 Waves

Travelling waves

As we saw in Section 7 the case of the transverse oscillation of individual masses the time dependence of the normal modes oscillation is sinusoidal. Let us consider the case that the time dependence of the vibrating string at $x=0$ is also sinusoidal, $y(x,0)=\sin(\omega t)$. In this case, from Eq. (9.9), the full x,t dependence is given by the "wave"

$$y(x,t)=A\sin(kx+\omega t)+B\sin(kx-\omega t) \tag{10.1}$$

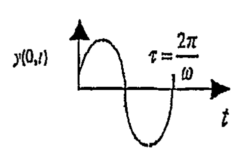
where A, B and k are constants. The speed of the wave is

$$c=\frac{\omega}{k} \tag{10.2}$$

Its frequency, f, is inversely proportional to its period, τ, and is given by

$$f=\frac{1}{\tau}=\frac{\omega}{2\pi} \tag{10.3}$$

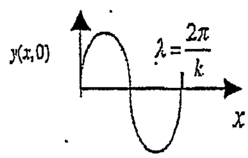
This is illustrated in the figure



Finally its wavelength, λ, is inversely proportional to its "wavenumber", k, and is given by

$$\lambda = \frac{2\pi}{k} \qquad (10.4)$$

as is clear from the figure showing $y(x,0)$



We can write the equation of a travelling wave in a number of analogous forms:

	Velocity	Wavelength	Period	Angular frequency
$A\sin(kx-\omega t)$	ω/k	$2\pi/k$	$2\pi/\omega$	ω
$A\sin k(x-vt)$	v	$2\pi/k$	$2\pi/vk$	vk
$A\sin\left[2\pi\left(\frac{x}{\lambda}-\frac{t}{\tau}\right)\right]$	λ/τ	λ	τ	$2\pi/\tau$
$A\sin\left[2\pi(x-vt)/\lambda\right]$	v	λ	λ/v	$2\pi v/\lambda$

Note that it is often more convenient to represent a travelling wave by a complex exponential (this is particularly useful when one wants to combine phases):

$$y(x,t) = \mathrm{Re}\left\{A\exp\left[i(kx-\omega t)\right]\right\} = \mathrm{Re}\left\{|A|\exp\left[i(kx-\omega t+\phi)\right]\right\} \qquad (10.5)$$

where A is complex, $A = |A|e^{i\phi}$.

Sometimes it is more convenient to switch x and t, *i.e.*

$$y(x,t) = A\sin(\omega t - kx) \qquad (10.6)$$

This is still a travelling wave moving to the right.

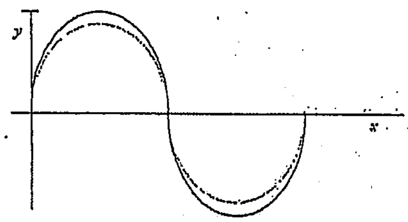
Of course, as discussed above, for a non-sinusoidal wave moving to right with speed c, we can always write it as $f(x-\mathrm{v}t)$ for some (non-sinusoidal) function f.

Stationary waves

Consider a string with two waves of equal amplitude travelling in the opposite directions

$$\begin{aligned} y &= A\sin(kx-\omega t) + A\sin(kx+\omega t) \\ &= 2A\sin kx\cos\omega t \end{aligned} \qquad (10.7)$$

Consider now the displacement at some fixed t, e.g. $t = 0$, for which $y = 2A\sin kx$. Some small time δt later the displacement becomes $y = 2A\cos(\omega.\delta t)\sin kx$. This has *exactly* the same x-dependence, and has *not* shifted at all (zeroes of y stay at the same x) but the amplitude now just a bit smaller.



Hence as t increases, the wave stays in the same place, but the amplitude varies.

e.g.

at $\omega t = \pi/2$ $\quad y = 0$ everywhere

at $\omega t = \pi$ $\quad y(x, \pi/\omega) = -y(x, 0)$

This wave is called a "Stationary Wave". It can be written in several forms, *e.g.* $y = 2A\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi t}{\tau}$, and there may be arbitrary (constant) phases, *e.g.* $2A\sin(kx + \phi_1)\cos(\omega t + \phi_2)$.

Longitudinal and Transverse Waves and Polarisation

A wave on a string has a displacement perpendicular to the (x) direction in which the wave travels *i.e.* it is a "Transverse" wave. But a sound wave has molecules moving backwards and forwards along the direction of wave propagation, *i.e.* sound waves are "Longitudinal" waves. Another example of a longitudinal wave is a coiled spring with the compression moving along spring.

Now clearly there is only one direction along the direction of propagation but there are two directions perpendicular to the direction of propagation *i.e.* a transverse wave moving along the $x-$axis can have two directions of transverse displacement, $y = A\sin(kx - \omega t)$ and $z = B\sin(kx - \omega t + \phi)$. Hence transverse waves can be "polarised", but longitudinal waves can not. Polarisation states correspond to definite values for the amplitudes A and B:

A	B	ϕ	Polarisation state	
1	0	-	Linear	→
0	1	-	Linear	↑
1	1	0	Linear	↗
1	1	π	Linear	↖
1	1	$\pi/2$	Circular	(LH)
1	1	$-\pi/2$	Circular	(RH)

10.0 Partial differential equations

The wave equation (8.5) is a differential equation for $y(t, x)$ containing partial derivatives of second order, linear and homogeneous. The transformation used in Eq. (9.2) to find its solution corresponds to curves in the t, x plane which are called the characteristics of the equation. This section gives a brief introduction to basic concepts on partial differential equations (PDE) and the method of characteristics. The subject will be developed in 2nd-year math methods.

A partial differential equation (PDE) is a differential equation for a function of several variables $u(x_1, \ldots, x_n)$, and contains partial derivatives of u with respect to $x_1, \ldots, x_n$. The order of the PDE is the order of the highest partial derivative appearing in it. As in the case of ordinary differential equations (ODE), a PDE can be given in terms of a differential operator D,

$$Du = h \ .$$

The function $h = h(x_1, \ldots, x_n)$ is a given function of the independent variables. If $h = 0$ the PDE is homogeneous; if $h \neq 0$ the PDE is inhomogeneous. A PDE is linear if the differential operator D is linear, i.e., if it obeys

$$D(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 D u_1 + \alpha_2 D u_2$$

for any two functions u_1, u_2 and any two constants α_1, α_2.

Consider for instance a case with two variables, $u(x, y)$. Denote partial derivatives as $\partial u / \partial x \equiv u_x$, $\partial^2 u / \partial x^2 \equiv u_{xx}$, and so forth. So a first-order linear PDE can be written in the general form

$$A(x, y) u_x + B(x, y) u_y + C(x, y) u = R(x, y)$$

where A, B, C, and R are given functions of x and y. If $R = 0$ the PDE is homogeneous. The equation

$$u u_{xx} + u_y^2 - u^2 = 0$$

is an example of a nonlinear second-order PDE.

Analogously to general solutions of ODEs depending on arbitrary integration constants, general solutions of PDEs depend on arbitrary functions. The D'Alembert solution (9.9) of the wave equation is an example of this. In order to determine such functions, one needs to prescribe the values of the function u and/or its derivatives along the boundary of a given region. Different kinds of boundary conditions are appropriate for different PDEs. An example of boundary value problem for the wave equation is considered in Sec. 9.2. It is an example of Cauchy boundary conditions, to which we come back later in this section.

In the following we consider second-order linear PDEs. These encompass a great number of equations relevant for physics applications.

A. Second-order linear PDEs

In the case of two variables x, y a second-order linear PDE has the general form

$$A(x, y) u_{xx} + 2B(x, y) u_{xy} + C(x, y) u_{yy} + D(x, y) u_x + E(x, y) u_y + F(x, y) u = R(x, y) \ , \quad (1)$$

where A, B, C, D, E, F, and R are given functions of x and y. It is useful to classify these equations, based on the coefficients of second-order derivatives, according to sgn $(B^2 - AC)$ as follows:

i) $B^2 - AC < 0$: elliptic; an example is the Laplace equation $u_{xx} + u_{yy} = 0$.

ii) $B^2 - AC = 0$: parabolic; an example is the diffusion equation, or heat equation, $u_t - \alpha u_{xx} = 0$.

iii) $B^2 - AC > 0$: hyperbolic; an example is the wave equation $u_{tt} - c^2 u_{xx} = 0$.

So for instance electrostatics problems are governed by elliptic equations; transport phenomena by parabolic equations; wave propagation by hyperbolic equations.

Since it only depends on the 2nd-order coefficients, this classification is valid also for quasi-linear equations, namely equations that are linear with respect to the highest partial derivatives, in this case 2nd-order.

The above classification is given in general point by point. We will consider examples in which the coefficients are constant, so that the type of the equation is the same for every point.

Different types of boundary conditions are appropriate to the different types of equations above. Cauchy boundary conditions consist in prescribing the values of the function u and of its normal derivative $u_n \equiv \partial u / \partial n$ on a given curve γ in the xy plane. We find that Cauchy conditions are of primary importance for hyperbolic PDEs. There exist other important types of boundary conditions, which are weaker than Cauchy: Dirichlet conditions, in which only the values of the function u are prescribed, and Neumann conditions, in which only the values of the normal derivative u_n are prescribed. Both are relevant for elliptic and parabolic equations.

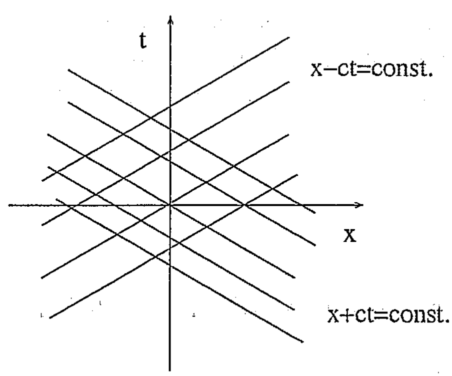
B. Characteristics

We have obtained D'Alembert's solution (9.9) of the wave equation by making the transformation (9.2) to new variables $x \mp ct$. The curves in the xt plane

$$x - ct = \text{const.} \,,$$

$$x + ct = \text{const.} \,,$$

play a special role for the wave equation and are called the *characteristics* of the wave



equation. We here introduce the basic concepts and results on characteristics. Consider the 2nd-order linear PDE in Eq. (1),

$$Au_{tt} + 2Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = R\ .$$

It is convenient to introduce the matrix of the 2nd-order coefficients

$$Q = \begin{pmatrix} A & B \\ B & C \end{pmatrix} .$$

Characteristics of the above PDE are defined as the curves

$$\chi(t,x) = \text{const.} \tag{2}$$

such that their normal n either is rotated by 90° by Q or is annihilated by Q, i.e.,

$$n \cdot Q\ n = 0\ .$$

Since the normal n is proportional to the gradient $\nabla\chi = (\chi_t, \chi_x)$, we have

$$\nabla\chi \cdot Q\ \nabla\chi = 0\ .$$

We can write this condition explicitly as

$$(\chi_t \quad \chi_x) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \chi_t \\ \chi_x \end{pmatrix} = 0\ ,$$

that is,

$$A\chi_t^2 + 2B\chi_t\chi_x + C\chi_x^2 = 0\ . \tag{3}$$

Example. For the wave equation

$$-\frac{1}{c^2}\, u_{tt} + u_{xx} = 0\ ,$$

the matrix Q of the 2nd-order coefficients is given by

$$Q = \begin{pmatrix} -1/c^2 & 0 \\ 0 & 1 \end{pmatrix} .$$

The curves

$$\chi_{\mp}(t,x) = x \mp ct = \text{const.}$$

are the characteristics of the wave equation because

$$\nabla\chi_{\mp} = \begin{pmatrix} \mp c \\ 1 \end{pmatrix} \Rightarrow \nabla\chi \cdot Q\, \nabla\chi = (\mp c \quad 1) \begin{pmatrix} -1/c^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mp c \\ 1 \end{pmatrix} = (\mp c \quad 1) \begin{pmatrix} \pm 1/c \\ 1 \end{pmatrix} = 0\ .$$

It may also be useful to rewrite Eq. (3) in terms of the derivative $x'(t)$ of the function $x = x(t)$ defined implicitly by Eq. (2). This is given by[1]

$$x'(t) = -\frac{\chi_t}{\chi_x} \ .$$

Then by substituting $\chi_t = -\chi_x x'$ into Eq. (3) we get

$$A[x'(t)]^2 - 2Bx'(t) + C = 0 \ . \tag{6}$$

This quadratic equation in x' illustrates that hyperbolic PDEs ($B^2 - AC > 0$) have 2 families of characteristics, parabolic PDEs ($B^2 - AC = 0$) have 1 ($Q \ \nabla\chi = 0$), and elliptic PDEs ($B^2 - AC < 0$) have none.

So for instance there are no characteristics for the Laplace equation $u_{xx} + u_{yy} = 0$. In the example above of the wave equation $u_{tt} - c^2 u_{xx} = 0$ we identified the two characteristics $x \mp ct =$ const. The diffusion equation $u_t - \alpha u_{xx} = 0$ has one family of characteristics $t =$ const.

Characteristics $\chi_\mp(t, x) =$ const. can be used to solve hyperbolic equations by means of the transformation of variables

$$\eta = \chi_-(t, x),$$
$$\xi = \chi_+(t, x).$$

An example is in D'Alembert solution of the wave equation in Sec. 9. Characteristics also serve to analyze whether boundary value problems are well-defined. An example of this is discussed in the next section.

C. Cauchy boundary conditions and characteristics

In this section we consider the Cauchy boundary value problem for a 2nd-order linear PDE and ask under which conditions this is well-posed. The main result is that Cauchy

[1] To see this, rewrite Eq. (2) as

$$g(t, x) \equiv \chi(t, x) - \text{const.} = 0 \ , \tag{4}$$

and Taylor-expand to first order the function of two variables $g(t, x)$ about an arbitrary point t_0, x_0 on the curve,

$$g(t, x) = g(t_0, x_0) + g_t(t_0, x_0)(t - t_0) + g_x(t_0, x_0)(x - x_0) + \dots \ ,$$

where the dots stand for quadratic terms in the Taylor expansion. Now using Eq. (4) we have

$$\frac{x - x_0}{t - t_0} = -\frac{g_t(t_0, x_0)}{g_x(t_0, x_0)} + \dots = -\frac{\chi_t(t_0, x_0)}{\chi_x(t_0, x_0)} + \dots \tag{5}$$

Taking the limit $t \to t_0$ in Eq. (5), the left hand side gives the derivative $x'(t_0)$, while the higher order terms in the right hand side give vanishing contribution. Thus

$$x'(t) = -\frac{\chi_t}{\chi_x} \ .$$

boundary conditions on curve γ are well-defined provided γ is not a characteristic (Cauchy-Kowalevska theorem).
We consider the 2nd-order linear PDE in Eq. (1), which has the form

$$Au_{tt} + 2Bu_{tx} + Cu_{xx} = H(u_t, u_x, u, t, x) \ , \tag{7}$$

where we have written explicitly the 2nd-order terms, and incorporated the lower order terms and inhomogeneous term in the function H.
Let us consider Cauchy boundary conditions for $u(t,x)$ on a given curve γ. That is, let us suppose that, given the curve γ specified by

$$G(t,x) = 0 \ , \tag{8}$$

where G is a given function of t and x, the values of the function u and of its derivative along the normal direction n to γ, $u_n = n \cdot \nabla u$, are assigned on γ.
The normal and tangential directions to γ, n and τ, are respectively proportional to the gradient of G and orthogonal to it. So $n = (G_t, G_x)/|\nabla G|$, $\tau = (-G_x, G_t)/|\nabla G|$. Given u on γ, we can compute the tangential derivative $u_\tau = \tau \cdot \nabla u$. From u_n and u_τ we can obtain u_t and u_x. In order for the boundary value problem to be well-defined, we have to be able to determine higher order derivatives as well, so that from the differential equation and the boundary conditions we can fully calculate the function u.
Consider a second derivative in the tangential direction:

$$|\nabla G| \ \frac{\partial}{\partial \tau} \ u_t = |\nabla G| \ \tau \cdot \nabla \ u_t = \begin{pmatrix} -G_x & G_t \end{pmatrix} \begin{pmatrix} u_{tt} \\ u_{xt} \end{pmatrix} = -G_x u_{tt} + G_t u_{xt} \tag{9}$$

$$|\nabla G| \ \frac{\partial}{\partial \tau} \ u_x = |\nabla G| \ \tau \cdot \nabla \ u_x = \begin{pmatrix} -G_x & G_t \end{pmatrix} \begin{pmatrix} u_{tx} \\ u_{xx} \end{pmatrix} = -G_x u_{tx} + G_t u_{xx} \tag{10}$$

We can treat Eqs. (7),(9),(10) as three linear equations in u_{tt}, u_{tx}, u_{xx}, having one and only solution if the determinant is nonzero, i.e., if

$$\det \begin{pmatrix} A & 2B & C \\ -G_x & G_t & 0 \\ 0 & -G_x & G_t \end{pmatrix} \neq 0 \ .$$

This implies

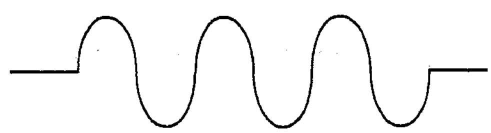
$$AG_t^2 + 2BG_tG_x + CG_x^2 \neq 0 \ .$$

By comparison with Eq. (3), we see that the curve γ specified by Eq. (8) must not be a characteristic in order for the Cauchy conditions to be well-defined.
The case of the wave equation with the initial conditions at $t = 0$ in Eqs. (9.10),(9.11) is an example in which Cauchy conditions on the function and on its normal derivative are given on a curve, the horizontal line $t = 0$, which does not belong to the families of characteristics.
In the case of the diffusion equation $u_t - \alpha u_{xx} = 0$, on the other hand, as noted below Eq. (6) the curves $t =$ const. are characteristics. So we do not expect Cauchy type of conditions at $t = 0$ to give a well-defined boundary value problem in this case.

11 Group and Phase velocity.

11.1 Information transmission

Let us consider how a signal might be sent via a wave. It is necessary to modulate the wave otherwise it will convey zero information. An example is given by the signal shown below



Here the wave is *on* for time T, then *off* i.e. $y = A\sin\left(kx - \omega t\right)$ for $\left|kx - \omega t\right| \leq \omega T/2$ and $y = 0$ for $\left|kx - \omega t\right| > \omega T/2$. It is important to note that this is *not* a single frequency wave for which $y = A\sin\left(kx - \omega_0 t\right)$ for some frequency ω_0 and which would apply for **all** $kx - \omega_0 t$. In order to buld up a wave which is *off* at some time it is necessary to have a superposition of a range of frequencies.

[This actually requires FOURIER TRANSFORMS which we won't meet until the 2nd year, but we will give a simple illustration of the result below]

11.2 Group and phase velocity

Let us consider in more detail our example of a wave that conveys information which is made up of a superposition of many waves with a range of frequencies. For illustration we consider a superposition of a discrete number of waves.

$$y(x,t) = \sum_{n=1}^{N} D_n \cos\left(k_n x - \omega_n t\right) \tag{11.1}$$

where D_n are constants. An immediate question is how fast does envelope carrying the signal move? The answer to this is called the "group velocity". This can be quite different from the answer to the question how fast do the individual waves in the superposition travel? This is called the "phase velocity".

Let us first compute these velocities.

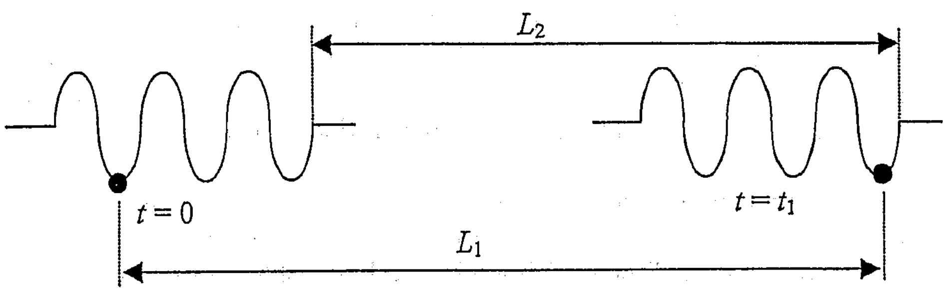


Figure 11.1

From Figure 11.1 we may determine the group velocity, g, by working out how far the front of the signal moves in time t_1. This is given by:

$$g = \frac{L_2}{t_1} \tag{11.2}$$

What about the phase velocity? In Eq. (11.1) the individual waves have phase velocity $v_n = \frac{\omega_n}{k_n}$.

Non-dispersive medium

If these are all equal, $v_n = v$, the envelope moves with unchanged shape and the phase velocity is the same as the group velocity. That this is the case follows from the fact that in this case Eq. (11.1) may be rewritten as

$$y(x,t) = \sum_{n=1}^{N} D_n \cos(k_n x - \omega_n t) = \sum_{n=1}^{N} D_n \cos k_n (x - vt) = y(x - vt) \tag{11.3}$$

i.e. a function of $(x - vt)$ only. This is the form of d'Alembert's solution obtained in Section 9.2 and shows that the whole wave moves with the phase velocity, v, with unchanged shape, i.e. the group velocity equals the phase velocity.

Dispersive medium

Now let us consider waves in a "dispersive medium", *i.e.* one in which different frequencies are transmitted at different speeds:

$$y = A\sin(kx - \omega t) \text{ where } v = \omega / k = f(\omega) \tag{11.4}$$

A well known example is the passage of light through a glass prism where different colours emerge at different angles because the refractive index μ (= c/v) depends on the colour (ω), i.e. v depends on ω.

What happens to our example in Eq. (11.1)? In this case the individual phase velocities $v_n = \frac{\omega_n}{k_n}$ are not all equal and the group velocity can be quite different from the phase velocity. This is illustrated in Figure 11.1 where one phase velocity, given by $v_p = \frac{L_1}{t_1}$, is greater than the group velocity g. To make this clearer let us turn to a more definite, albeit oversimplified, model of a wave packet of the form of Eq. (11.1) but with $N = 2$, i.e. a superposition of just two waves.

11.4 A simple approach to building a wave packet

Although we really need an infinite number of different frequency waves to construct a finite wave packet here we will illustrate the result using just two waves, y_1 and y_2.

$$\begin{aligned} y_1 &= A\sin\left[(k + \delta k)x - (\omega + \delta\omega)t\right] \\ y_2 &= A\sin\left[(k - \delta k)x - (\omega - \delta\omega)t\right] \end{aligned} \tag{11.5}$$

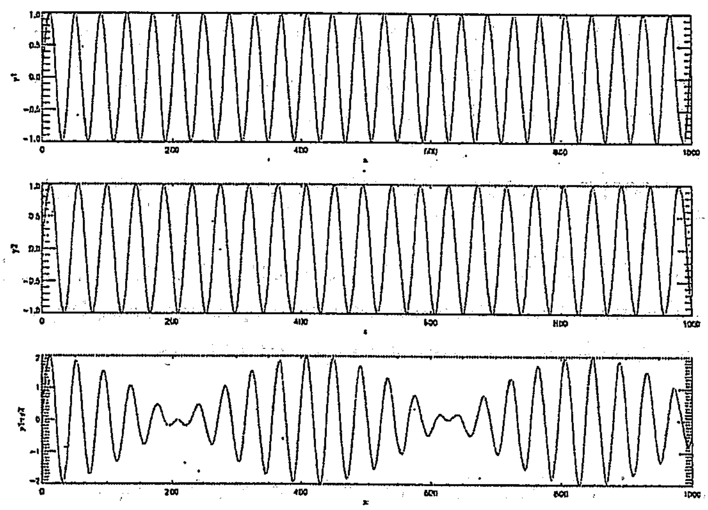
where δk and $\delta\omega$ are small. Then

$$y = y_1 + y_2 = 2A\cos(\delta k.x - \delta\omega.t)\sin(kx - \omega t) \tag{11.6}$$

The first term has a very long wavelength $2\pi/\delta k$ (and very long period $2\pi/\delta\omega$). This describes the slowly varying envelope which moves at speed $g = \delta\omega/\delta k$.

The second term is very similar to y_1 and y_2 and has speed $v = \omega/k$ for the individual wavelets.

The addition of these waves gives the following



Plot for y_1,y_2 and y_1+y_2 where $\delta k/k = 0.05$.

This is not exactly one packet of waves, but an infinite series of sausages (because we used two waves instead of infinite number). However the essential point is that envelope moves with speed $g = \delta\omega/\delta k$ and not at the mean phase velocity $v = \omega/k$.

11.5 Group velocity for a complete wave packet

For the case of a real wave packet of finite extent comprised of an infinite number of different frequency waves the group velocity is given by

$$g = \frac{d\omega}{dk} \tag{11.7}$$

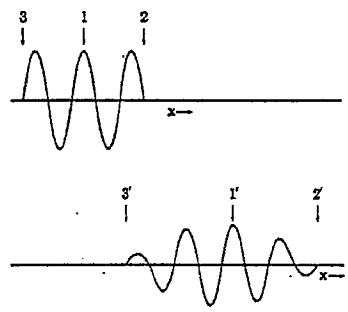
For the case of a dispersive medium with $v = \omega / k = f(\omega)$ then $\frac{d\omega}{dk}$ need not equal $\frac{\omega}{k}$ and so the group velocity need not equal the phase velocity. As we have discussed it is the group velocity that determines the speed a signal can propagate and this is constrained by the theory of relativity to be less than or equal to the speed of light, $g \leq c$. However the phase velocity, v, can readily be greater than c.

We can illustrate this with a simple example. Consider an experiment to determine the speed of light by measuring the time of flight of a pulse of light through a long tube filled with air to determine its velocity, v_0. Since $\mu \neq 1$, we need to correct the measured time to determine the speed of light, c, in vacuum. The naïve answer is $c = \mathrm{v}_0\ \mu$ but this does not give the group velocity if μ depends on k i.e. on the frequency.

To make this explicit let us assume the phase velocity, v, varies with colour as $\mathrm{v} = \frac{\omega}{k} = c\left(1-bk\right)$, *i.e.* $\mu = \frac{c}{\mathrm{v}} = \frac{1}{1-bk}$. Then $\omega = c\left(k - bk^2\right)$ giving for the group velocity $g = \frac{\partial\omega}{\partial k} = c\left(1-2bk\right)$, different from the phase velocity, $\mathrm{v}=\frac{c}{\mu}$. From this one sees it is incorrect to change the measured velocity v_0 to $\mathrm{v}_0\mu$ to allow for the effect of the air.

11.6 Dispersion and the spreading of the wave packet

Since a finite wave packet involves a superposition of waves with a range of frequencies, if the velocity is frequency dependent then, necessarily, the shape of the packet will change because the individual wave components are moving with



different speed. This is illustrated above. However this immediately demonstrates a problem to measuring the group velocity because if the shape of the wave packet is changing it is impossible precisely to measure the envelope's speed. This effect may be seen algebraically from the fact that the group velocity $g = \frac{d\omega}{dk}$ is in general a function of the frequency and thus not uniquely determined for a wave packet comprised of a range of frequencies. In practice this may not be so important for a long wave packet as in this case the range of frequencies in the packet is quite small so the ambiguity of the group velocity is also small.

11.7 Alternative expressions for g

There are many equivalent expressions for the group velocity. Above we used

$$g = \frac{d\omega}{dk} \tag{11.8}$$

But $\omega = \mathrm{v}k$ so we can write it as

$$g = \mathrm{v} + k\frac{d\mathrm{v}}{dk} \tag{11.9}$$

Further, since $k = 2\pi/\lambda$ we also have

$$g = \mathrm{v} - \lambda\frac{d\mathrm{v}}{d\lambda} \tag{11.10}$$

Yet another form is given using $\mathrm{v} = c/\mu$

$$g = \frac{c}{\mu}\left(1 + \frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right) \qquad (11.11)$$

i.e. $g \neq c/\mu$ in a dispersive medium as we found in our example above.

Note here we have used k and λ as measured in a medium. More conventionally we use the wavelength, λ', measured in vacuum. In terms of it we have

$$g = \mathrm{v}\left(1 - \frac{1}{1 + \frac{\mathrm{v}}{\lambda'}\frac{d\lambda'}{dv}}\right)$$

since $\lambda' = \lambda\frac{c}{\mathrm{v}}$ and $f = \frac{c}{\lambda'} = \frac{\mathrm{v}}{\lambda}$.

Examples

1. Consider a wave traveling through a dispersive medium in which the relation between wave speed v and wavelength λ is given by

$$v^2 = c^2 + \lambda^2\omega_0^2 ,$$

where c and ω_0 are constant parameters. Show that the product of phase and group velocities equals c^2.

By differentiating the above relation we have

$$2v\ dv = 2\lambda\ d\lambda\ \omega_0^2 , \qquad i.e. \quad \frac{dv}{d\lambda} = \frac{\lambda\omega_0^2}{v} .$$

Then the group velocity is given by

$$g = v - \lambda\frac{dv}{d\lambda} = v - \frac{\lambda^2\omega_0^2}{v} = v - \frac{v^2 - c^2}{v} = \frac{c^2}{v} .$$

Thus

$$gv = c^2 .$$

2. A model for describing propagation of waves in deep water is

$$v = C/\sqrt{k} ,$$

where C is constant. Show that the group velocity equals half the phase velocity.

Since $v = \omega/k$, we have $\omega/k = C/\sqrt{k}$, i.e.,

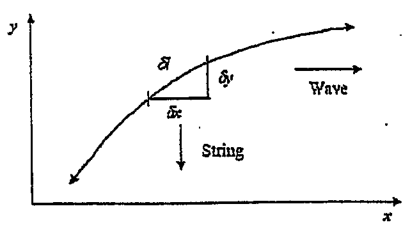
$$\omega = C\sqrt{k}\,.$$

Therefore the group velocity is

$$g = \frac{d\omega}{dk} = \frac{C}{2\sqrt{k}} = \frac{1}{2}\,v\,.$$

12 Energy of vibrating string

Let us assume the string carries a transverse wave $y = A\sin(kx - \omega t)$ and consider a small portion of the string as shown in the figure below. The string has linear density ρ. There are two contributions to the energy, the kinetic energy and the potential energy.



Kinetic energy (K.E.)

The K.E. of the segment shown is given by

$$\begin{aligned} K.E. &= \frac{1}{2}\rho\delta x\left(\frac{\partial y}{\partial t}\right)^2 \\ &= \frac{1}{2}\rho A^2\omega^2\cos^2(kx - \omega t)\,\delta x \end{aligned} \qquad (12.1)$$

We may now integrate over a distance l that contains a whole number of wavelengths, at fixed t.

$$K.E. = \frac{1}{2}\rho A^2\omega^2\int_x^{x+l}\cos^2(kx - \omega t)\,dx = \frac{1}{2}\rho A^2\omega^2 \times \frac{l}{2} \qquad (12.2)$$

Hence the K.E. per unit length is given by

$$K.E./l = \frac{1}{4}\rho A^2\omega^2 \tag{12.3}$$

Potential energy

The wave stretches the string leading to an increase in its potential energy relative to the value in its equilibrium position. The potential energy in a the segment is given by the work done in stretching the string. The tension, T, is the force resisting the stretching so we have

$$\begin{aligned}\text{P.E. of segment } &= T\left(\delta l - \delta x\right)\\ &= T\delta x\left(\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}-1\right)\\ &\approx \frac{1}{2}TA^2k^2\cos^2\left(kx-\omega t\right)\delta x\end{aligned} \tag{12.4}$$

Integrating this over l leads to the result

$$P.E./l = \frac{1}{4}TA^2k^2 \tag{12.5}$$

How do the K.E. and P.E. compare? Since $\text{v} = \omega / k = \sqrt{T/\rho}$ we have $Tk^2 = \rho\omega^2$ and hence

$$\text{P.E.}/l = \frac{1}{4}TA^2k^2 = \frac{1}{4}\rho A^2\omega^2 = \text{K.E.}/l \tag{12.6}$$

Hence the total energy per unit length is given by

$$\text{E}/l = \frac{1}{2}\rho A^2\omega^2 \tag{12.7}$$

12.1 Energy flow

In time t, the wave moves a distance vt (note that since we are considering a plane wave of definite frequency it is the phase velocity, v, that is relevant). Hence the energy flow/unit time, F, is given by

$$\begin{aligned}F &= \left(\frac{1}{2}\rho A^2\omega^2\right)vt/t\\ &= \frac{1}{2}\rho\omega^2A^2v\\ &= \frac{1}{2}\rho\omega^3A^2/k\\ &= \frac{1}{2}Tk^2A^2v\\ &= \frac{1}{2}T\omega kA^2\end{aligned} \tag{12.8}$$

(Note that the energy flow may also be calculated by considering the rate at which the string to the left of a position does work on the right.

$$\begin{aligned} E &= F_y v_y \\ &= -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \end{aligned} \tag{12.9}$$

Substituting for y and differentiating we find the same answer.)

13 Solution of the wave equation – separation of variables

In this section we introduce the method of separation of variables for finding solutions to the wave equation. In Section 8 we derived the wave equation which for convenience we repeat here

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{13.1}$$

where v is the speed of the wave.

We now look for solutions that have the "separated" form:

$$y(x,t) = X(x)T(t) \tag{13.2}$$

Substituting this into the wave equation we find:

$$T(t)\frac{d^2X(x)}{dx^2} = \frac{1}{v^2}X(x)\frac{d^2T(t)}{dt^2} \tag{13.3}$$

or

$$\frac{\ddot{X}}{X} = \frac{1}{v^2}\frac{\ddot{T}}{T} \tag{13.4}$$

Now the left-hand side of this equation is a function of just x, while the right-hand side is a function of just t. The only way that a function of x can equal a function of t for all x and t is if both are equal to a *constant*, C_S. We look for a solution in which this constant, known as the "separation constant", is *negative*, $C_S = -k^2$. i.e.

$$\frac{\ddot{X}}{X} = \frac{1}{v^2}\frac{\ddot{T}}{T} = C_S = -k^2 \tag{13.5}$$

Then we find that

$$\begin{aligned} \ddot{X} + k^2X &= 0 \\ \ddot{T} + k^2v^2T &= 0 \end{aligned} \tag{13.6}$$

The separation constant reduces the partial differential equation to two ordinary differential equations which may be solved using standard methods to give:

$$\begin{aligned} X &= A\cos kx + B\sin kx \\ T &= C\cos kvt + D\sin kvt \end{aligned} \tag{13.7}$$

where A, B, C and D are unknown constants which may be found from the boundary conditions. If the boundary conditions constrain $A = 0$, and $D = 0$ then we find that

$$y(x,t) = X(x)T(t) = B'\sin kx\cos kvt \tag{13.8}$$

(B'=BC) which is a standing wave of the form that we introduced earlier in Eq. (10.7) (N.B. $kv=\omega$). Eq. (10.7) shows the connection of the solution we have just obtained by separation of variables with that we obtained previously by d'Alembert's solution.

In deriving this solution we chose our separation constant to be negative. What happens if we had chosen it to be positive? If $C_S = +k^2$ then we find:

$$y(x,t) = X(x)T(t) = \left(Ae^{kx} + Be^{-kx}\right)\left(Ce^{kvt} + De^{-kvt}\right) \qquad (13.9)$$

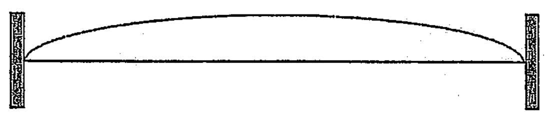
If $C_S = 0$ then we find:

$$y(x,t) = X(x)T(t) = (A + Bx)(C + Dt) \qquad (13.10)$$

and the list continues for complex separation constants. Which of these solutions is elevant depends on the physical situation. In these lectures we are concerned with the sinusoidally varying solutions that correspond the negative choice for the separation variable.

Even having chosen the sign the solution is not unique because any value of k may be used. Indeed, since the differential equation is linear, the principle of superposition applies and a linear combination of any number of solutions with different values of k will still be a solution. In the next section we will see how this can be used to find general solutions to various physical situations.

13.1 Wave on a string with fixed ends



Consider a string given an initial displacement, $y(x,0)$, and then released. What happens subsequently? Since the string has fixed ends we also have the boundary conditions $y(0,t) = y(L,t) = 0$.

We have found by separation of variables that a solution to the wave equation is given by:

$$y(x,t) = (A\cos kx + B\sin kx)(C\cos kvt + D\sin kvt) \qquad (13.11)$$

As we shall see we can use a linear superposition of such solutions with different choices for k to build a solution that satisfies the boundary conditions. Let us consider the boundary conditions in turn.

i) The string is initially at rest, *i.e.* $\partial y / \partial t = 0$ for all x, which requires $D = 0$
ii) $y(0,t) = 0$ requires that $A = 0$
iii) $y(L,t) = 0$ requires that $kL = n\pi$, where n is any integer. This latter condition is known as the eigenvalue equation and it limits k to be an integer multiple of $n\pi / L$; each of these values corresponds to a normal mode.

With this we can write the most general solution consistent with the boundary conditions i)-iii) as a linear superposition of the normal mode solutions:

$$y(x,t) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi vt}{L} \qquad (13.12)$$

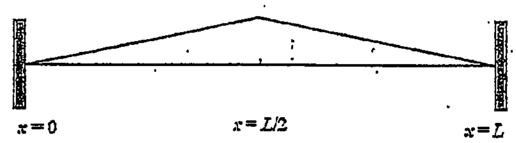
So far we have not imposed the final initial condition, namely the initial displacement $y(x,0)$. If the initial string displacement initially corresponds to a normal mode

$$y(x,0) = B\sin\frac{m\pi x}{L} \tag{13.13}$$

then by comparision with Eq. (13.12), $A_n = B$ for $n = m$, $A_n = 0$ otherwise and the subsequent motion is given by

$$y(x,t) = B\sin\frac{m\pi x}{L}\cos\frac{m\pi vt}{L}$$

What if initial displacement does not correspond to a normal mode?
e.g. a plucked guitar string with shape



In this case the subsequent motion described by an infinite sum of normal modes as in Eq. (13.12). At $t=0$

$$y(x,0) = \sum_n A_n \sin\frac{n\pi x}{L} \tag{13.14}$$

and the form of the initial displacement will determine the A_n coefficients. In the second year you will study Fourier series of this type. It turns out that the coefficients can easily be found due to the orthogonality of the sine functions over the range $0 \le x < L$, giving

$$A_n = \frac{2}{L}\int_0^L y(x,0)\sin\frac{n\pi x}{L}dx \tag{13.15}$$

However, since this lies beyond the scope of these lectures, let us consider a simple case in which the initial distribution is given by

$$y(x,0) = \sin\frac{\pi x}{L} + \frac{1}{2}\sin\frac{2\pi x}{L} \tag{13.16}$$

Comaring with Eq. (13.12) we see that at subsequent times:

$$y(x,t) = \sin\frac{\pi x}{L}\cos\frac{\pi vt}{L} + \frac{1}{2}\sin\frac{2\pi x}{L}\cos\frac{2\pi vt}{L} \tag{13.17}$$

Note that in this case we have a superposition of two normal modes and, unlike the case with just a single normal mode, the subsequent motion is not equal to the initial displacement × varying amplitude. Moreover since the shorter wavelengths oscillate faster the shape of the wave varies during oscillation.

13.2 Normal modes and energy

In the previous subsection we have determined the normal modes of the string of length L with fixed ends:

$$k_n = \frac{n\pi}{L}, \quad n \text{ integer },$$

corresponding to the normal frequencies

$$\omega_n = vk_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}} .$$

The n-th normal mode can then be written as

$$y_n(x,t) = A\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}+\delta\right) . \tag{18}$$

The general solution can be expressed as a linear superposition of the normal modes. The amplitude A and phase δ are to be determined from the initial conditions at $t=0$ (e.g., we have seen that if $\partial y/\partial t = 0$ at $t=0$ for all x, then $\delta = 0$).
We can now ask what is the energy associated with a given normal mode, and how the total energy is distributed among different modes. To see this, we use the expressions found in Sec. 12 for the kinetic energy density dK/dx per unit length and potential energy density dU/dx per unit length. From Eqs. (12.1) and (12.4) we have

$$\frac{dK}{dx} = \frac{1}{2}\,\rho\,\left(\frac{\partial y}{\partial t}\right)^2 , \tag{19}$$

$$\frac{dU}{dx} = \frac{1}{2}\,T\,\left(\frac{\partial y}{\partial x}\right)^2 . \tag{20}$$

Then the kinetic energy for the n-th normal mode is obtained by using Eq. (18) and integrating Eq. (19) over the length of the string:

$$\begin{aligned}
K_n &= \frac{1}{2}\,\rho\int_0^L dx\,\left(\frac{\partial y_n}{\partial t}\right)^2 \\
&= \frac{1}{2}\rho A^2\left(\frac{n\pi v}{L}\right)^2\sin^2\left(\frac{n\pi vt}{L}+\delta\right)\int_0^L dx\,\sin^2\left(\frac{n\pi x}{L}\right) \\
&= \frac{\rho A^2 n^2\pi^2 v^2}{4L}\sin^2\left(\frac{n\pi vt}{L}+\delta\right) .
\end{aligned}$$

Similarly, the potential energy for the n-th normal mode is given by

$$\begin{aligned}
U_n &= \frac{1}{2}\,T\int_0^L dx\,\left(\frac{\partial y_n}{\partial x}\right)^2 \\
&= \frac{1}{2}TA^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi vt}{L}+\delta\right)\int_0^L dx\,\cos^2\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

$$= \frac{TA^2n^2\pi^2}{4L} \cos^2\left(\frac{n\pi vt}{L} + \delta\right) .$$

Since $\rho v^2 = T$, the total energy for the n-th normal mode is

$$E_n = K_n + U_n = \frac{\rho A^2 n^2 \pi^2 v^2}{4L} = \frac{\rho L A^2 \omega_n^2}{4} .$$

To find the total energy of the system, we express $y(x,t)$ as a linear superposition of the normal modes:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L} + \delta\right) .$$

When we evaluate the kinetic and potential energies K and U, similarly to what we have just done for a single normal mode, we have to differentiate and square this $\sum$, and then integrate over x. Upon integration, all the contributions from the crossed terms arising from squaring the sum vanish, because

$$\int_0^L dx \ \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = 0 \qquad \text{if } n \neq m .$$

Consequently,

$$K = \sum_n K_n \quad , \quad U = \sum_n U_n ,$$

and the total energy of the system is given by the sum of the energies of each normal mode,

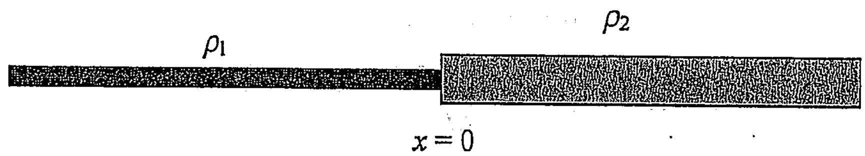
$$E = \sum_n E_n .$$

14 Wave reflection at a boundary

In optics reflection is caused at a boundary separating regions with different refractive indices in which the light travels at different speeds. Exactly the same phenomena occurs for the transverse waves propagating on a string. In this case on obtains a boundary separating regions with different wave speeds by joining two strings of different linear densities, $\rho_{1,2}$. Since the tension, T, remains the same across the boundary, the phase velocities are different

$$v_{1,2} = \sqrt{T / \rho_{1,2}} \tag{14.1}$$

This is illustrated in the diagram



$x < 0$
Lighter string, larger v
→ Incident wave
← Reflected wave

$x > 0$
Heavier string, smaller v
→ Transmitted wave

Incident wave	$A \sin(\omega t - k_1 x)$
Reflected wave	$A' \sin(\omega t + k_1 x)$
Transmitted wave	$A'' \sin(\omega t - k_2 x)$

Note:

1) We have made a slight change in the convention using $\omega t - kx$ instead of $kx - \omega t$. This is not crucial, but agrees with other treatments.
2) All waves have the same ω [This follows from the boundary conditions at $x = 0$. These cannot be satisfied for all t unless ω is constant – see below]
3) The transmitted wave has $-k_2 x$ (right mover)
4) The reflected wave has $+k_1 x$ (left-mover)

The amplitudes A' and A'' are determined in terms of the incident amplitude A from the boundary conditions at $x = 0$. There are two boundary conditions

A) $y(-\varepsilon, t) = y(+\varepsilon, t)$ where ε is a number close to zero. This just says the string is continous.

B) $\frac{\partial y}{\partial x}(-\varepsilon, t) = \frac{\partial y}{\partial x}(+\varepsilon, t)$. This follows because, for small angular displacements, the vertical component of the force on the left of the boundary $T\frac{\partial y}{\partial x}(-\varepsilon, t)$ must be balanced by the vertical component of the force on the left of the boundary $T\frac{\partial y}{\partial x}(\varepsilon, t)$. (For small displacements the horizontal component of the force vanishes to the order considered here – see the discussion above Eq. (7.1))

These boundary conditions are more commonly written as

$$\begin{aligned} y_1(0,t) &= y_2(0,t) \\ \frac{\partial y_1}{\partial x}(0,t) &= \frac{\partial y_2}{\partial x}(0,t) \end{aligned} \tag{14.2}$$

where $y_1(x,t) = A\sin(\omega t - k_1 x) + A'\sin(\omega t + k_1 x)$ is the sum of the incident and reflected waves on the left of the boundary and $y_2(x,t) = A''\sin(\omega t - k_2 x)$ is the transmitted wave on the right of the boundary. Thus we have

$$A\sin\omega t + A'\sin\omega t = A''\sin\omega t \quad \Rightarrow \quad A + A' = A'' \qquad (14.3)$$

and

$$-k_1 A\cos\omega t + k_1 A'\cos\omega t = -k_2 A''\cos\omega t \quad \Rightarrow \quad k_1\left(A - A'\right) = k_2 A'' \qquad (14.4)$$

These may be solved to give

$$r = \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$
$$t = \frac{A''}{A} = \frac{2k_1}{k_1 + k_2} \qquad (14.5)$$

where r and t are known as the *amplitude reflection* and *amplitude transmission* coefficients respectively.

Special cases:

1) $k_1 = k_2 \quad \Rightarrow A' = 0,\ t = \frac{A''}{A} = 1$ No reflection

2) $k_1 < k_2 \quad \Rightarrow A'$ is negative

In this case the reflected wave may be written as

$-|A'|\sin(\omega t + k_1 x) = |A'|\sin(\omega t + k_1 x + \pi)$ i.e. there is a phase change at a **rare-dense** boundary (since $v = \omega / k = \sqrt{T/\rho}$, $k_1 < k_2$ implies $\rho_1 < \rho_2$).

3) $k_1 > k_2 \quad \Rightarrow A'$ is positive

4) $\rho_2 \to \infty \quad \Rightarrow k_2 \to \infty$ hence $r = \frac{A'}{A} \to -1$. In this case the tension $T \to 0$ and there is no wave in the very heavy string.

14.1 The energy flux at the boundary

In Eq. (12.8) we showed that the energy flux is proportional to the (amplitude)^2. It is thus very tempting, but **wrong** to think that the power reflection and transmission coefficients should be just $R_P = r^2$, and $T_P = t^2$. That this cannot be true can immediately be seen from the fact that $R_p + T_p \neq 1$ which would correspond to non-conservation of energy. To see this, note that from Eq. (14.5)

$$r^2 + t^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 + \left(\frac{2k_1}{k_1 + k_2}\right)^2 = \frac{5k_1^2 - 2k_1 k_2 + k_2^2}{\left(k_1 + k_2\right)^2} \neq 1$$

The origin of the error is clear from Eq.(12.8) which expresses the power flux as $P=\frac{1}{2}T\omega kA^2$. Although T and ω are constant within the string k is different on each side of the boundary, explaining the error. Hence at the boundary between the two strings:

Incident power flux $\quad P_I=\frac{1}{2}T\omega k_1A^2$

Reflected power flux $\quad P_R=\frac{1}{2}T\omega k_1A'^2$

Transmitted power flux $\quad P_T=\frac{1}{2}T\omega k_2A''^2$

Hence the power reflection and transmission coefficients are given by

$$R_P=\frac{P_R}{P_I}=\frac{A'^2}{A^2}=r^2=\left(\frac{k_1-k_2}{k_1+k_2}\right)^2 \tag{14.6}$$

$$T_P=\frac{P_T}{P_I}=\frac{k_2A''^2}{k_1A^2}=\frac{k_2}{k_1}t^2=\frac{k_2}{k_1}\left(\frac{2k_1}{k_1+k_2}\right)^2=\frac{4k_1k_2}{\left(k_1+k_2\right)^2} \tag{14.7}$$

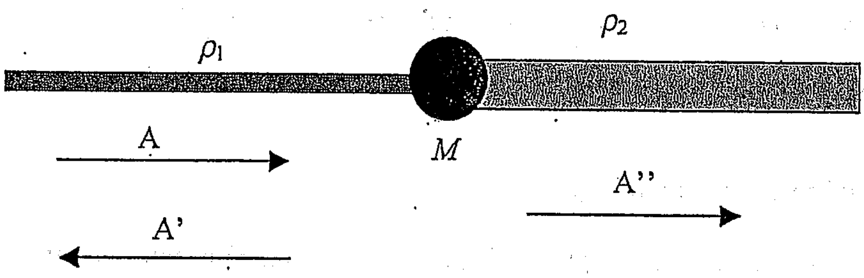
Hence

$$R_P+T_P=\left(\frac{k_1-k_2}{k_1+k_2}\right)^2+\frac{k_2}{k_1}\left(\frac{2k_1}{k_1+k_2}\right)^2=\frac{k_1^2+2k_1k_2+k_2^2}{\left(k_1+k_2\right)^2}=1 \tag{14.8}$$

as required by energy conservation.

14.2 Reflection from a mass at the boundary

Suppose that a finite mass M is fixed at the boundary between two semi-infinite pieces of string of density ρ_1 and ρ_2:



The string is clearly continuous and hence the first boundary condition is

$$y_1(0,t)=y_2(0,t) \tag{14.9}$$

as before. The second boundary condition however is not the same since we now have a finite mass at the boundary. In this case the sum of the forces at the boundary act on the mass and generates its acceleration in the transverse direction according to Newton's 2nd law:

$$-T\frac{\partial y_1}{\partial x}(0,t)+T\frac{\partial y_2}{\partial x}(0,t)=M\frac{\partial^2 y_1}{\partial t^2}(0,t)=M\frac{\partial^2 y_2}{\partial t^2}(0,t) \tag{14.10}$$

Eqs. (14.9) and (14.10) are the boundary conditions for this problem. In this case we see that they involve both first and second derivatives and for this reason it is easiest to use a complex exponential representation for the waves i.e.

$$
\begin{aligned}
y_1(x,t) &= \mathrm{Re}\left\{A\exp\left(i\left(\omega t - k_1 x\right)\right)\right\} + \mathrm{Re}\left\{A'\exp\left(i\left(\omega t + k_1 x\right)\right)\right\} \\
y_2(x,t) &= \mathrm{Re}\left\{A''\exp\left(i\left(\omega t - k_2 x\right)\right)\right\}
\end{aligned}
\qquad (14.11)
$$

where A is real but A' and A'' may be complex. Inserting this in Eqs. (14.9) and (14.10) gives

$$A + A' = A'' \qquad (14.12)$$

and

$$ik_1 TA - ik_1 TA' - ik_2 TA'' = -\omega^2 M\left(A + A'\right) = -\omega^2 MA'' \qquad (14.13)$$

which simplifies to

$$ik_1\left(A - A'\right) = \left(ik_2 - \omega^2 M / T\right)A'' \qquad (14.14)$$

From Eqs. (14.12) and (14.14) we can determine the amplitude reflection and amplitude transmission coefficients:

$$r = \frac{A'}{A} = \frac{\left(k_1 - k_2\right)T - i\omega^2 M}{\left(k_1 + k_2\right)T + i\omega^2 M} \equiv |r|e^{i\phi_r} \qquad (14.15)$$

$$t = \frac{A''}{A} = \frac{2k_1 T}{\left(k_1 + k_2\right)T + i\omega^2 M} \equiv |t|e^{i\phi_t} \qquad (14.16)$$

Substituting this in Eq. (14.11) gives the real amplitudes with the reflected and transmitted waves having phase shifts ϕ_r and ϕ_t relative to the incident wave.

Consider the special case where the second line has zero mass per unit length, i.e. we just have a mass on the end of a line. i.e. $k_2 = 0$. Then:

$$r = \frac{A'}{A} = \frac{k_1 T - i\omega^2 M}{k_1 T + i\omega^2 M} \qquad (14.17)$$

Hence if $M = 0$, $r = 1$ and if M is large, $r = -1 = e^{i\pi}$.

15 Characteristic Impedance

Although more commonly used for cases of electromagnetic waves travelling in transmission lines or space, the concept of "characteristic Impedance" may actually be defined for *any* wave motion and is a useful descriptive parameter.

For transverse waves on a string the characteristic impedance Z is defined as the force acting in the y-direction divided by the velocity of the string in the y-direction, *i.e.*

$$Z = \frac{F_y}{v_y} = \frac{-T\frac{\partial y}{\partial x}}{\frac{\partial y}{\partial t}} \tag{15.1}$$

For a sine wave travelling in the positive x-direction $y(x,t) = A\sin(kx - \omega t)$ and thus the characteristic impedance is:

$$Z = \frac{Tk}{\omega} = \frac{T}{v} = (T\rho)^{1/2}$$

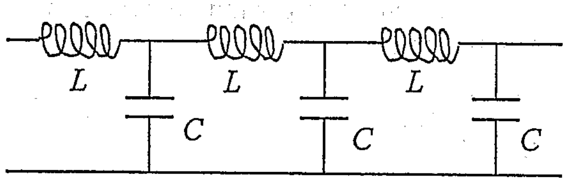
We may express the reflection and transmission coefficients of Eqs. (14.15) and (14.16) in terms of impedances

$$\begin{aligned} r &= \frac{A'}{A} = \frac{(k_1 - k_2)T - i\omega^2 M}{(k_1 + k_2)T + i\omega^2 M} = \frac{Z_1 - (Z_2 + Z_M)}{Z_1 + (Z_2 + Z_M)} \\ t &= \frac{A''}{A} = \frac{2k_1 T}{(k_1 + k_2)T + i\omega^2 M} = \frac{2Z_1}{Z_1 + Z_2 + Z_M} \end{aligned} \tag{15.2}$$

where we have substituted the wavenumbers for the characteristic impedances $Z_{1,2} = \frac{Tk_{1,2}}{\omega}$ and also for the 'impedance' of the mass: $Z_M = i\omega M$.

16 Other Waves

16.1 Waves on an Electrical Line



The voltage change across the inductor of self-inductance L in one of the elements is given (via Faraday's law) as:

$$L\frac{\partial I}{\partial t} = -\delta V = -\frac{\partial V}{\partial x}\delta x \tag{16.1}$$

where we have assumed that the elements are so small that the voltage change can be related to the variation of voltage with distance.

Similarly the current flowing though the capacitance C in each element is given by:

$$\frac{\partial Q}{\partial t} = C\frac{\partial V}{\partial t} = -\delta I = -\frac{\partial I}{\partial x}\delta x \tag{16.2}$$

Hence dividing these two equations by δx gives:

$$\frac{L}{\delta x}\frac{\partial I}{\partial t} = L'\frac{\partial I}{\partial t} = -\frac{\partial V}{\partial x} \tag{16.3}$$

and

$$\frac{C}{\delta x}\frac{\partial V}{\partial t} = C'\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x} \tag{16.4}$$

where C' is the capacitance per unit length of the transmission line and L' is the self-inductance per unit length.

Differentiating Eq. (16.3) with respect to t and differentiating Eq. (16.4) with respect to x gives:

$$\frac{\partial^2 V}{\partial t \partial x} = -\frac{1}{C'}\frac{\partial^2 I}{\partial x^2} = -L'\frac{\partial^2 I}{\partial t^2} \tag{16.5}$$

Hence:

$$\frac{\partial^2 I}{\partial x^2} = L'C'\frac{\partial^2 I}{\partial t^2} \tag{16.6}$$

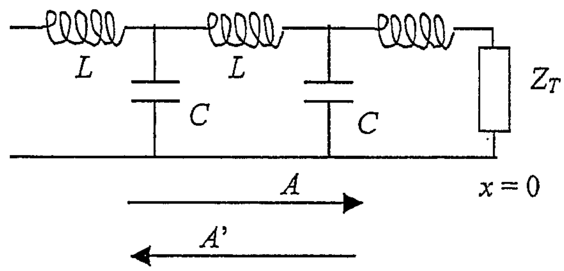
which is a wave equation for I. Similarly we can obtain a wave equation for V:

$$\frac{\partial^2 V}{\partial x^2} = L'C'\frac{\partial^2 V}{\partial t^2} \tag{16.7}$$

Hence the electrical line supports waves with a phase speed $v = 1/\sqrt{L'C'}$.

Suppose we have a voltage wave travelling in a single direction: $V = V_0 \sin(\omega t - kx)$, then using the above equations we find that $I = (V_0 / Z)\sin(\omega t - kx)$ where Z is the characteristic impedance given by $Z = \sqrt{L'/C'}$.

Reflection at a terminated line



Voltage wave travels along a transmission line of characteristic impedance Z_0 and is partly reflected by a terminating impedance Z_T at x = 0. The voltage and current on the line are thus given by:

$$\begin{aligned} V &= A\exp\left(i\left(\omega t - kx\right)\right) + A'\exp\left(i\left(\omega t + kx\right)\right) \\ Z_0 I &= A\exp\left(i\left(\omega t - kx\right)\right) - A'\exp\left(i\left(\omega t + kx\right)\right) \end{aligned} \tag{16.8}$$

At $x = 0$, V and I are related by the terminating impedance and thus:

$$\frac{V}{Z_0 I} = \frac{Z_T}{Z_0} = \frac{A + A'}{A - A'} \tag{16.9}$$

Hence

$$r = \frac{A'}{A} = \frac{Z_T - Z_0}{Z_T + Z_0} \tag{16.10}$$

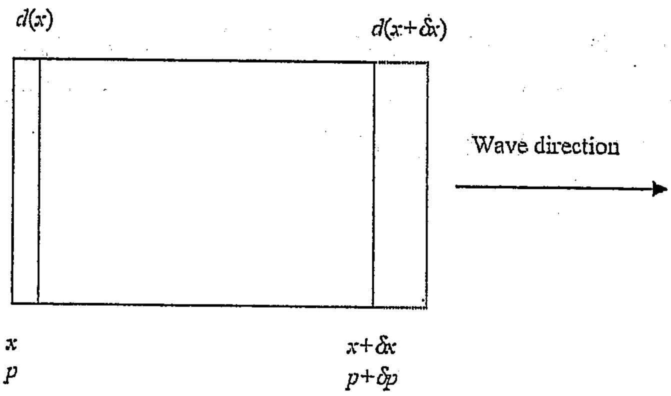
Hence

when $Z_T \to 0$, $r \to -1$
when $Z_T = Z_0$, $r = 0$
when $Z_T \to \infty$, $r \to +1$

These limits appear to be the reverse of what we found for the mass on the end of a string. However the characteristic impedance is defined differently. For strings $Z = Tk / \omega = T / \text{v}$. However for transmission lines $Z = L'\text{v}$ which is somewhat different !

16.2 Sound

Sound waves correspond to longitudinal waves associated with the compression of the medium. Consider waves propagating in the x-direction as shown in the Figure



The compression caused by the passage of the sound wave changes the pressure and the volume of the element of gas originally between x and $x+\delta x$. Consider a cross-sectional area A of the wave.

With no sound wave, the volume of the element of gas is $V_1 = A\delta x$. With the sound wave passing the new volume of the element is

$$\begin{aligned} V_2 &= A\left(d\left(x+\delta x\right) - d\left(x\right)\right) + V_1 \\ &\approx A\frac{\partial d}{\partial x}\delta x + V_1 \\ &= V_1\frac{\partial d}{\partial x} + V_1 \end{aligned} \tag{16.11}$$

Hence

$$\delta V = V_2 - V_1 = \frac{\partial d}{\partial x} V \tag{16.12}$$

The instantaneous pressure of the gas may be approximated by

$$\begin{aligned} p &\approx p_0 + \frac{\partial p}{\partial V}\delta V \\ &= p_0 + \frac{\partial p}{\partial V}\frac{\partial d}{\partial x}V \\ &= p_0 - K\frac{\partial d}{\partial x} \end{aligned} \tag{16.13}$$

where K is the bulk modulus of the gas (or solid) defined as:

$$K = -V\frac{\partial p}{\partial V} \tag{16.14}$$

The net force, F, acting on the element is:

$$\begin{aligned} F &= A\left[p(x) - p(x+\delta x)\right] \\ &= -\frac{\partial p}{\partial x}A\delta x \end{aligned} \tag{16.15}$$

From Newton's 2nd law

$$\rho\ A\ \delta x\ \frac{\partial^2 d}{\partial t^2} = F = -\frac{\partial p}{\partial x}A\ \delta x$$

Thus

$$\rho\ \frac{\partial^2 d}{\partial t^2} = -\frac{\partial p}{\partial x} \tag{16.16}$$

Differentiating Eq. (16.13) with respect to x gives

$$\frac{\partial p}{\partial x} = -K\ \frac{\partial^2 d}{\partial x^2}$$

Hence

$$\frac{\partial^2 d}{\partial x^2} = \frac{\rho}{K}\frac{\partial^2 d}{\partial t^2} \tag{16.17}$$

which is the wave equation with $v = \sqrt{\frac{K}{\rho}}$.

The characteristic impedance is defined as

$$Z = \frac{K\frac{\partial d}{\partial x}}{\frac{\partial d}{\partial t}} \tag{16.18}$$

so for a wave travelling in positive x-direction

$$Z = \frac{Kk}{\omega} = \frac{K}{v} = (\rho K)^{1/2} \tag{16.19}$$

Isothermal compressions: $PV = \text{constant} \Rightarrow K = -V\frac{\partial p}{\partial V} = p \Rightarrow v = \sqrt{p/\rho}$

Adiabatic compressions: $PV^{\gamma} = \text{constant} \Rightarrow K = -V\frac{\partial p}{\partial V} = \gamma p \Rightarrow v = \sqrt{\gamma p/\rho}$

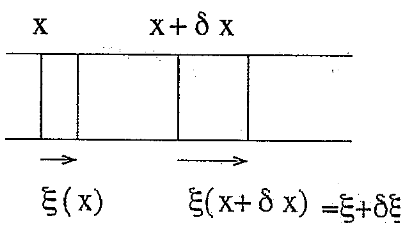
From kinetic theory we know $p = \frac{1}{3}\rho\overline{v^2}$ where here v is the molecular speed. Hence

$v_{sound} = \sqrt{\frac{\gamma}{3}\overline{v^2}}$ and thus $v_{sound} \approx v_{rms}$ of molecules since sound is transmitted by moving molecules.

N.B. For longitudinal waves travelling along solid bars, we get very similar solutions except that the bulk modulus K must be replaced by Young's modulus Y. We see this next.

16.3 Longitudinal oscillations in a solid bar

We consider the longitudinal vibrations of an elastic bar. For the thin slice between x and $x+\delta x$ in the figure, let ξ be the longitudinal displacement at x, and $\xi+\delta\xi$ the longitudinal displacement at $x+\delta x$. Let A be the cross sectional area. Let ρ be the mass density.



The ratio of the elongation $\delta\xi$ to the original length δx defines the strain. For small enough strain, the force per unit area is given according to Hooke's law by Young's elasticity modulus Y times the strain. Then from Newton's 2nd law we have

$$\rho\ A\ \delta x\ \frac{\partial^2\xi}{\partial t^2} = F_{|x+\delta x} - F_{|x} = A\ Y\ \left[\frac{\partial\xi}{\partial x}(x+\delta x) - \frac{\partial\xi}{\partial x}(x)\right] \ .$$

By expanding the right hand side in powers of δx, we obtain the wave equation

$$\frac{\partial^2\xi}{\partial x^2} = \frac{1}{v^2}\,\frac{\partial^2\xi}{\partial t^2} \quad , \quad v = \sqrt{Y/\rho} \ .$$

Example

Study the longitudinal oscillations of an elastic bar of mass density ρ, Young's modulus Y and length L with one fixed end (i.e., no displacement) and one free end (i.e., no strain). Determine the normal frequencies.

The boundary conditions require that the longitudinal displacement $\xi(x,t)$ satisfies

$$\xi(0,t) = 0 \quad , \qquad \partial\xi/\partial x(L,t) = 0 \ .$$

Let us write solutions of the wave equation for $\xi(x,t)$ in the separated-variable form

$$\xi(x,t) = \left(A \sin \frac{\omega x}{v} + B \cos \frac{\omega x}{v}\right) \cos(\omega t + \phi) ,$$

where $v = \sqrt{Y/\rho}$. Then the conditions at $x = 0$ and $x = L$ imply

$$\xi(0,t) = 0 \Rightarrow B = 0 ,$$

$$\frac{\partial \xi}{\partial x}(L,t) = 0 \Rightarrow \cos \frac{\omega L}{v} = 0 \Rightarrow \frac{\omega L}{v} = \frac{\pi}{2} + n\pi .$$

Thus the normal frequencies are given by

$$\omega_n = \frac{\pi v}{L}(n + \frac{1}{2}) = \frac{\pi}{L}\sqrt{\frac{Y}{\rho}}(n + \frac{1}{2}) \qquad (n \text{ integer}) .$$

The lowest normal mode frequency is

$$\omega_0 = \frac{\pi}{2L}\sqrt{\frac{Y}{\rho}} .$$

16.4 Sound waves in a 3-dimensional case

In Sec. 16.2 we have considered sound waves in a one-dimensional case. In general in a compressible gas sound waves propagate in all three directions. We here extend the treatment of Sec. 16.2 to the three-dimensional case.
For each point $\mathbf{x} = (x, y, z)$ we have displacements in three directions, d_i, $i = 1, 2, 3$. These form a "displacement field", $\mathbf{d} = (d_1, d_2, d_3)$, function of $\mathbf{x}$ and t. Each component d_i satisfies an equation of motion of the form (16.16),

$$\rho \, \ddot{d_i} = -\partial p / \partial x_i , \qquad \text{i.e.,} \quad \rho \, \ddot{\mathbf{d}} = - \text{ grad } p . \tag{20}$$

The variation of volume due to the sound wave is given by the 3-d generalization of Eqs. (16.11),(16.12). We have

$$\Delta V = \Delta y \Delta z[d_1(x + \Delta x, y, z) - d_1(x, y, z)] + \Delta z \Delta x[d_2(x, y + \Delta y, z) - d_2(x, y, z)]$$

$$+\Delta x \Delta y[d_3(x, y, z + \Delta z) - d_3(x, y, z)] = \Delta x \Delta y \Delta z \left(\frac{\partial d_1}{\partial x} + \frac{\partial d_2}{\partial y} + \frac{\partial d_3}{\partial z}\right) = V \text{ div } \mathbf{d}$$

Using the bulk modulus K (16.14), we express the variation of pressure with volume as

$$p = p_0 + \frac{\partial p}{\partial V} \, \Delta V = p_0 + \left(-\frac{K}{V}\right) \, (V \text{ div } \mathbf{d}) = p_0 - K \text{ div } \mathbf{d} .$$

Therefore

$$\frac{\partial p}{\partial x_i} = -K \frac{\partial}{\partial x_i} \operatorname{div} \mathbf{d} \quad , \qquad \text{i.e.,} \quad \operatorname{grad} p = -K \operatorname{grad} \operatorname{div} \mathbf{d} \quad . \tag{21}$$

Inserting Eq. (21) into Eq. (20) we obtain

$$\rho \, \ddot{\mathbf{d}} = K \operatorname{grad} \operatorname{div} \mathbf{d} \quad . \tag{22}$$

Eq. (22) is the equation for sound waves in a three-dimensional compressible gas. The displacement field $\mathbf{d}$ can be expressed in terms of scalar potential φ

$$\mathbf{d} = \operatorname{grad} \varphi \, .$$

Then φ obeys the wave equation

$$\nabla^2 \varphi - \frac{1}{v^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad , \qquad v = \sqrt{K/\rho} \quad ,$$

where

$$\nabla^2 = \operatorname{div} \operatorname{grad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad .$$