SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

- more than 1 unknown function: $y_1(x), y_2(x), \ldots, y_n(x)$
- set of ODEs that couple $y_1, \ldots, y_n$

▷ physical applications: systems with more than 1 degree of freedom. dynamics couples differential equations for different variables.

Example. System of first-order differential equations:

$$y_1' = F_1(x, y_1, y_2, \ldots, y_n)$$

$$y_2' = F_2(x, y_1, y_2, \ldots, y_n)$$

$$\cdots$$

$$y_n' = F_n(x, y_1, y_2, \ldots, y_n)$$

An nth-order differential equation

$$y^{(n)} = G(x, y, y', y'', \ldots, y^{(n-1)})$$

can be thought of as a system of n first-order equations.

- Set new variables $y_1 = y$; $y_2 = y'$; ... ; $y_n = y^{(n-1)}$
    - Then the system of first-order equations

$$y_1' = y_2$$

$$\ldots$$

$$y_{n-1}' = y_n$$

$$y_n' = G(x, y_1, y_2, \ldots, y_n)$$

is equivalent to the starting nth-order equation.

$\diamondsuit$ Systems of linear ODEs with constant coefficients can be solved by a generalization of the method seen for single ODE:

$$\text{General solution} \; = \; \mathrm{PI} \; + \; \mathrm{CF}$$

$\triangleright$ Complementary function CF by solving *system of auxiliary equations*

$\triangleright$ Particular integral PI from a *set of trial functions* with functional form as the inhomogeneous terms

Example (from June 2007 Prelims Paper)

The variables $\psi(z)$ and $\phi(z)$ obey the simultaneous differential equations

$$3\,\frac{d\phi}{dz} + 5\psi = 2z$$

$$3\,\frac{d\psi}{dz} + 5\phi = 0.$$

Find the general solution for ψ.

- Differentiating 2nd equation wrt z gives

$$0 = 3\,\frac{d^2\psi}{dz^2} + 5\,\frac{d\phi}{dz} = 3\,\frac{d^2\psi}{dz^2} + \frac{5}{3}\,(2z - 5\psi)$$

$$\text{i.e.}\quad \frac{d^2\psi}{dz^2} - \frac{25}{9}\,\psi = -\frac{10}{9}\,z$$

- Solve this 2nd-order linear ODE with constant coefficients:

CF: Auxiliary equation $m^2 - 25/9 = 0 \;\Rightarrow\; m_\pm = \pm 5/3$.

So CF $= Ae^{5z/3} + Be^{-5z/3}$.

PI: Trial function $\psi_0 = C_1 z + C_0 \;\Rightarrow$

$$-(25/9)C_1 z - (25/9)C_0 = -10/9z$$

$$\Rightarrow\; C_1 = 2/5\;,\; C_0 = 0 \;\Rightarrow\; \mathrm{PI} = 2z/5.$$

General solution for ψ :

$$\psi(z) = \mathrm{CF} + \mathrm{PI} = A\; e^{5z/3} + B\; e^{-5z/3} + \frac{2}{5}\; z\;.$$

Systems of linear equations with constant coefficients and driving terms

Ex 1

$$\frac{dx}{dt}+\frac{dy}{dt}+y=t,$$
$$-\frac{dy}{dt}+3x+7y=e^{2t}-1.$$

Known variables on RHS

$$L\begin{pmatrix}x\\y\end{pmatrix}\equiv\begin{pmatrix}\frac{d}{dt} & +\frac{d}{dt}+1\\ 3 & -\frac{d}{dt}+7\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{dx}{dt} & +\frac{dy}{dt}+y\\ 3x & -\frac{dy}{dt}+7y\end{pmatrix}.$$

$$L\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}t\\e^{2t}-1\end{pmatrix}$$

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ e^{2t} - 1 \end{pmatrix}$$

Complementary function :

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Set $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X\mathrm{e}^{\alpha t} \\ Y\mathrm{e}^{\alpha t} \end{pmatrix}$ α, X, Y complex nos to be determined

(In this case we will find α is real so X,Y will be taken real too)

$$L\begin{pmatrix} X\mathrm{e}^{\alpha t} \\ Y\mathrm{e}^{\alpha t} \end{pmatrix} \equiv \begin{pmatrix} \dfrac{\mathrm{d}\mathrm{e}^{\alpha t}}{\mathrm{d}t} & +\dfrac{\mathrm{d}\mathrm{e}^{\alpha t}}{\mathrm{d}t} + \mathrm{e}^{\alpha t} \\ 3\mathrm{e}^{\alpha t} & -\dfrac{\mathrm{d}\mathrm{e}^{\alpha t}}{\mathrm{d}t} + 7\mathrm{e}^{\alpha t} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \quad \begin{pmatrix} \alpha & \alpha + 1 \\ 3 & 7 - \alpha \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha & \alpha+1 \\ 3 & 7-\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\begin{vmatrix} \alpha & \alpha+1 \\ 3 & 7-\alpha \end{vmatrix} = \alpha(7-\alpha) - 3(\alpha+1) = 0 \Rightarrow \alpha = 3 \ , \ \alpha = 1$$

The allowed values of α determine the allowed values of X/Y:

$$\alpha = 3 \Rightarrow 3X + 4Y = 0 \Rightarrow Y = -3X/4$$

$$\alpha = 1 \Rightarrow X + 2Y = 0 \Rightarrow Y = -X/2$$

Hence CF is

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -3/4 \end{pmatrix} e^{3t} + X_b \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} e^{t}$$

X_a, X_b constants

Particular Integral

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} & +\frac{dy}{dt} + y \\ 3x & -\frac{dy}{dt} + 7y \end{pmatrix} = \begin{pmatrix} t \\ e^{2t} - 1 \end{pmatrix}$$

To find PI we use trial functions

Polynomial part :

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -1 \end{pmatrix}$$

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X_0 + X_1 t \\ Y_0 + Y_1 t \end{pmatrix} = \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix}$$

$$\begin{pmatrix} X_1 + Y_1 + Y_1 t + Y_0 \\ 3(X_0 + X_1 t) - Y_1 + 7(Y_0 + Y_1 t) \end{pmatrix} = \begin{pmatrix} t \\ -1 \end{pmatrix}$$

$$Y_1 = 1 \qquad X_1 + Y_0 = -1$$

$$3X_0 + 7Y_0 = 0; \quad 3X_1 + 7Y_1 = 0$$

$$X_1 = -\tfrac{7}{3}$$

$$Y_0 = -1 + \tfrac{7}{3} = \tfrac{4}{3}$$

$$X_0 = -\tfrac{7}{3}Y_0 = -\tfrac{28}{9}$$

Particular Integral

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} & +\frac{dy}{dt} + y \\ 3x & -\frac{dy}{dt} + 7y \end{pmatrix} = \begin{pmatrix} t \\ e^{2t} - 1 \end{pmatrix}$$

Exponential part :

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{2t}$$

$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{2t}$$

$$2X + (2+1)Y = 0 \quad \Rightarrow \quad X = -\tfrac{3}{2}Y$$

$$3X + (-2+7)Y = 1 \quad \Rightarrow \quad (-\tfrac{9}{2}+5)Y = 1$$

$$X = -3$$

$$Y = 2$$

The full solution :

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} e^{3t} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix}$$

Initial conditions

The full solution :

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} e^{3t} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{28}{9} - \frac{7}{3} t \\ \frac{4}{3} + t \end{pmatrix}$$

The constants must be fixed by initial conditions e.g.

$$\begin{cases} \dot{x}(0) = -\frac{19}{3} \\ \dot{y}(0) = 3 \end{cases}$$

$$\begin{pmatrix} -\frac{19}{3} \\ 3 \end{pmatrix} = 3X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + 2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{7}{3} \\ 1 \end{pmatrix}$$

$$3X_a + X_b = 2$$

$$-\tfrac{9}{4} X_a - \tfrac{1}{2} X_b = -2 \quad \Rightarrow \quad X_a = \frac{-2}{-3/2} = \frac{4}{3}$$

$$X_b = 2 - 3X_a = -2$$

Number of integration constants.

Usually, but not always, equals the sum of the order of the simultaneous equations

…an example that breaks this rule …

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = t$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3x + 7y = \mathrm{e}^{2t}.$$

3 integration constants?

$$\frac{d}{dt}\left(\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = t\right) - \left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3x + 7y = \mathrm{e}^{2t}\right) \quad \Rightarrow \quad \frac{dy}{dt} - 3x - 7y = 1 - e^{2t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = t$$

$$-\frac{dy}{dt} + 3x + 7y = -1 + e^{2t}$$

$$\Rightarrow \quad \begin{pmatrix} \frac{d}{dt} & \frac{d}{dt} + 1 \\ 3 & -\frac{d}{dt} + 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ e^{2t} - 1 \end{pmatrix}$$

Actually only 2 integration constants needed!

An alternative method of solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = t \qquad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = t - \frac{\mathrm{d}y}{\mathrm{d}t} - y$$

$$\frac{dy}{dt} - 3x - 7y = 1 - e^{2t}$$

$$\frac{d}{dt}\left(\frac{dy}{dt} - 3x - 7y = 1 - e^{2t}\right) \quad \Rightarrow \quad \frac{d^2y}{dt^2} - 3\left(\frac{dx}{dt}\right) - 7\frac{dy}{dt} = -2e^{2t}$$

An alternative method of solution

$$\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}+y=t \qquad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=t-\frac{\mathrm{d}y}{\mathrm{d}t}-y$$

$$\frac{dy}{dt}-3x-7y=1-e^{2t}$$

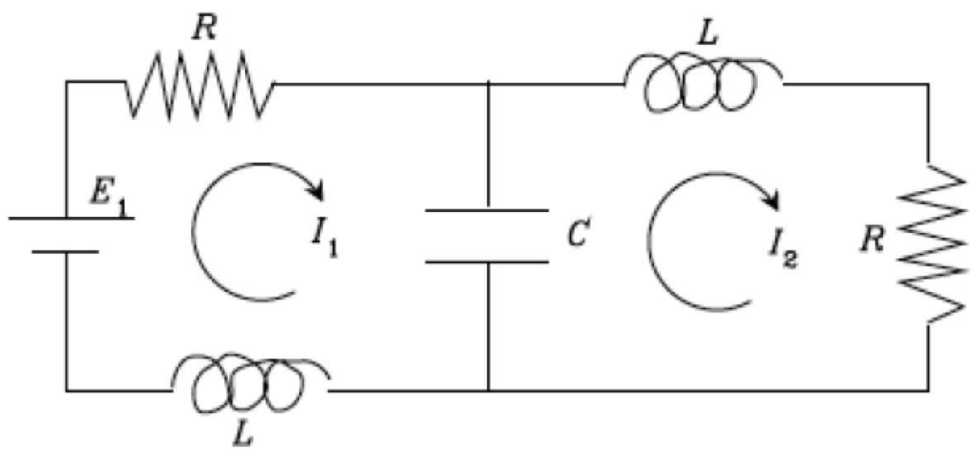
$$\frac{d}{dt}\left(\frac{dy}{dt}-3x-7y=1-e^{2t}\right) \Rightarrow \frac{d^2y}{dt^2}-3(t-\frac{dy}{dt}-y)-7\frac{dy}{dt}=-2e^{2t}$$

(Only 2 integration constants needed)

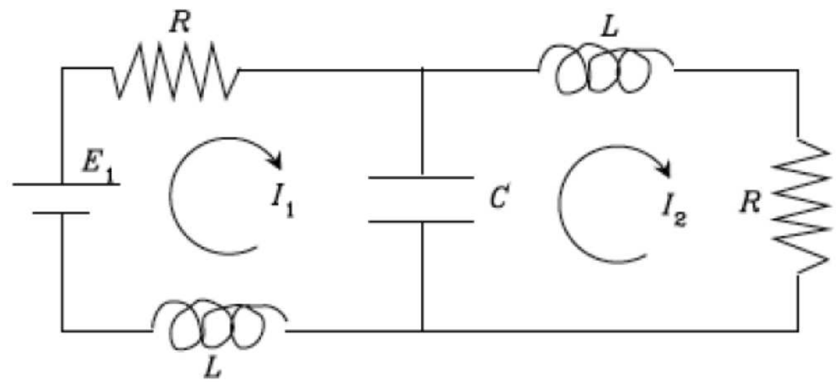
Ex 2 **LCR circuits**

The dynamics of a linear electrical circuit is governed by a system of linear equations with constant coefficients. These may be solved by the general matrix technique. In many cases they may be more easily solved by judicious addition and subtraction.

Ex 1



$$R^2 C < 8L$$



Using Kirchoff's laws :

$$RI_1 + \frac{Q}{C} + L\frac{\mathrm{d}I_1}{\mathrm{d}t} = E_1$$

$$L\frac{\mathrm{d}I_2}{\mathrm{d}t} + RI_2 - \frac{Q}{C} = 0.$$

Differentiate to eliminate Q

$$\frac{\mathrm{d}^2 I_1}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{1}{LC}(I_1 - I_2) = \frac{\mathrm{d}E_1}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}^2 I_2}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I_2}{\mathrm{d}t} - \frac{1}{LC}(I_1 - I_2) = 0.$$

$$\frac{d^2 I_1}{dt^2} + \frac{R}{L}\frac{dI_1}{dt} + \frac{1}{LC}(I_1 - I_2) = 0$$
$$\frac{d^2 I_2}{dt^2} + \frac{R}{L}\frac{dI_2}{dt} - \frac{1}{LC}(I_1 - I_2) = 0$$

$$\Rightarrow \begin{pmatrix} \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{\alpha t}$

$$\Rightarrow \begin{pmatrix} \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Auxiliary equations :

$$\alpha^2 + \frac{R}{L}\alpha = 0$$

$\alpha = 0, \quad -R/L$ Eigenvalues

$$\alpha^2 + \frac{R}{L}\alpha + \frac{2}{LC} = 0 \quad \Rightarrow \quad \alpha = -\tfrac{1}{2}\frac{R}{L} \pm \frac{i}{\sqrt{LC}}\sqrt{2 - \tfrac{1}{4}CR^2/L} = -\tfrac{1}{2}\frac{R}{L} \pm i\omega_R.$$

Eigenvalues

$$\frac{d^2 I_1}{dt^2} + \frac{R}{L}\frac{dI_1}{dt} + \frac{1}{LC}(I_1 - I_2) = 0$$

$$\frac{d^2 I_2}{dt^2} + \frac{R}{L}\frac{dI_2}{dt} - \frac{1}{LC}(I_1 - I_2) = 0$$

$$\Rightarrow \begin{pmatrix} \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{\alpha t}$

$$\Rightarrow \begin{pmatrix} \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equations :

$$\alpha^2 + \frac{R}{L}\alpha = 0$$

$\alpha = 0, \quad -R/L$ Eigenvalues

$$\begin{pmatrix} +\frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & +\frac{1}{LC} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} X \\ Y \end{pmatrix} = Ae^{i\phi} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Eigenvectors

$$\boxed{\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \left(A_0 + A_1 e^{-\frac{R}{L}t} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\frac{d^2 I_1}{dt^2} + \frac{R}{L}\frac{dI_1}{dt} + \frac{1}{LC}(I_1 - I_2) = 0$$
$$\frac{d^2 I_2}{dt^2} + \frac{R}{L}\frac{dI_2}{dt} - \frac{1}{LC}(I_1 - I_2) = 0$$

$$\Rightarrow \begin{pmatrix} \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{\alpha t}$

$$\Rightarrow \begin{pmatrix} \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equations :

$$\alpha^2 + \frac{R}{L}\alpha + \frac{2}{LC} = 0 \quad \Rightarrow \quad \alpha = -\tfrac{1}{2}\frac{R}{L} \pm \frac{i}{\sqrt{LC}}\sqrt{2 - \tfrac{1}{4}CR^2 / L} = -\tfrac{1}{2}\frac{R}{L} \pm i\omega_R.$$

$$\begin{pmatrix} -\frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & -\frac{1}{LC} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = Be^{i\phi} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigenvectors

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \mathrm{Re}\left\{ \begin{pmatrix} X \\ Y \end{pmatrix} e^{\alpha t} \right\} = Be^{-\frac{R}{2L}} \cos(\omega_R t + \phi) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$