We have seen

2nd-order linear ODEs with constant coefficients:

$$a_2 f'' + a_1 f' + a_0 f = h(x)$$

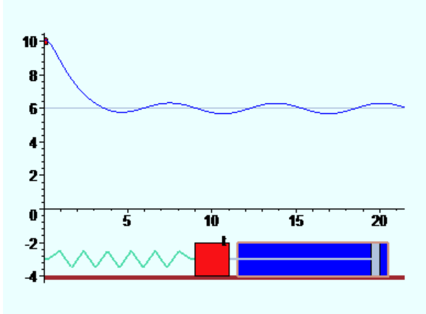
$\triangleright$ Complementary function CF by solving *auxiliary equation*

$\triangleright$ Particular integral PI by *trial function* with functional form of the inhomogeneous term

♠ Next: physical application to forced, damped oscillator

Oscillators

$$m\ddot{x} = -\underset{\text{spring}}{m\omega_0^2 x} - \underset{\text{friction}}{m\gamma\dot{x}} + \underset{\text{forcing}}{mF\cos\omega t}.$$



The associated complex equation is

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = F\mathrm{e}^{\mathrm{i}\omega t}.$$

Transients : CF - Auxiliary equation : $z = e^{\alpha t}$.

$\omega_0^2 - \frac{1}{4}\gamma^2 > 0$

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \quad \Rightarrow \quad \alpha = -\tfrac{1}{2}\gamma \pm \mathrm{i}\sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}$$

$$= -\tfrac{1}{2}\gamma \pm \mathrm{i}\omega_\gamma \quad \text{where} \quad \omega_\gamma \equiv \omega_0\sqrt{1 - \tfrac{1}{4}\gamma^2/\omega_0^2}.$$

Complementary function

Constant phase “shift”

$$x = \mathrm{e}^{-\gamma t/2}[A\cos(\omega_\gamma t) + B\sin(\omega_\gamma t)] = \mathrm{e}^{-\gamma t/2} N\cos(\omega_\gamma t + \psi)$$

Since $\gamma > 0$, the CF $\to 0$ as t $\to \infty$ .... CF describes "transients"

Steady state solutions

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = F e^{i\omega t}.$$

No damping exponential

Particular integral

$$x = \Re e\left(\frac{F e^{i\omega t}}{\omega_0^2 - \omega^2 + i\omega\gamma}\right).$$

i.e. the Particular integral describes the steady state solution after the transients have died away.

Since the denominator $= \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\, e^{i\phi}$ where $\phi \equiv \arctan\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)$ the particular integral can be written as

$$\boxed{x = \frac{F \Re e(e^{i(\omega t - \phi)})}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} = \frac{F\cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}}.$$

For $\phi > 0$, x achieves the same phase as F at t greater by $\left(\frac{\phi}{\omega}\right)$

- ϕ is called the "phase lag" of the response.

$$\boxed{x = \frac{F\,\Re(\mathrm{e}^{\mathrm{i}(\omega t-\phi)})}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} = \frac{F\cos(\omega t-\phi)}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}}$$

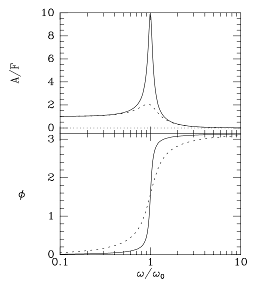
The amplitude of the response is $A = \dfrac{F}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$,

This has a maximum when $0 = \dfrac{\mathrm{d}A^{-2}}{\mathrm{d}\omega} \propto -4(\omega_0^2-\omega^2)\omega + 2\omega\gamma^2 \quad \Rightarrow \quad \omega^2 = \omega_0^2 - \tfrac{1}{2}\gamma^2.$

$\omega_R \equiv \sqrt{\omega_0^2-\gamma^2/2}$ is called the “resonant” frequency

The frictional coefficient causes the resonant frequency to be less than the normal frequency

$$\phi \equiv \arctan\left(\frac{\omega\gamma}{\omega_0^2-\omega^2}\right)$$



Oscillators

$$m\ddot{x} = -\underbrace{m\omega_0^2 x}_{\text{spring}} - \underbrace{m\gamma\dot{x}}_{\text{friction}} + \underbrace{mF\cos\omega t}_{\text{forcing}}.$$

$$\mathrm{F} = mF\cos\omega t, \qquad x = \frac{F\Re\mathrm{e}(\mathrm{e}^{\mathrm{i}(\omega t-\phi)})}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} = \frac{F\cos(\omega t-\phi)}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$

Power Input

(steady state) $\quad P = \frac{\partial W}{\partial t} = \mathrm{F}\dot{x} \qquad W = \int_{x(t_0)}^{x(t)} F\,dx'$

$$P = \mathrm{F}\dot{x} = mF\cos\omega t \times \frac{-F\omega\sin(\omega t-\phi)}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$

$$= \frac{\omega mF^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}[-\cos(\omega t)\sin(\omega t-\phi)]$$

$$= -\frac{\frac{1}{2}\omega mF^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}[\sin(2\omega t-\phi)+\sin(-\phi)].$$

Average over a period

$$\overline{P} = \frac{\frac{1}{2}\omega mF^2\sin\phi}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}.$$

Energy dissipated

$$m\ddot{x} = \underbrace{-\ m\omega_0^2 x}_{\text{spring}} \ \underbrace{-\ m\gamma\dot{x}}_{\text{friction}} \ \underbrace{+\ mF\cos\omega t}_{\text{forcing}}.$$

$$x = \frac{F\cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$\overline{D} = m\gamma\overline{\dot{x}\dot{x}} = \frac{m\gamma\omega^2 F^2 \frac{1}{2}}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}.$$

$$\overline{\sin^2(\omega t - \phi)} = \frac{1}{2}$$

$$\overline{D} = \overline{P} = \frac{\frac{1}{2}\omega m F^2 \sin\phi}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \quad \text{since} \quad \sin\phi = \gamma\omega / \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$\left(\phi \equiv \arctan\left(\tfrac{\omega\gamma}{\omega_0^2 - \omega^2}\right)\right)$$

Quality Factor

$$m\ddot{x} = \underbrace{-\ m\omega_0^2 x}_{\text{spring}} \ \underbrace{-\ m\gamma\dot{x}}_{\text{friction}} \ \underbrace{+\ mF\cos\omega t}_{\text{forcing}}.$$

Energy content of transient motion that the CF describes

$$E = \tfrac{1}{2}(m\dot{x}^2 + m\omega_0^2 x^2) \qquad x = e^{-\gamma t/2} A\cos(\omega_\gamma t + \psi)$$

$$= \tfrac{1}{2} mA^2 e^{-\gamma t}[\tfrac{1}{4}\gamma^2\cos^2\eta + \omega_\gamma\gamma\cos\eta\sin\eta + \omega_\gamma^2\sin^2\eta + \omega_0^2\cos^2\eta] \quad (\eta \equiv \omega_\gamma t + \psi)$$

$$E \simeq \tfrac{1}{2} m(\omega_0 A)^2 e^{-\gamma t} \qquad \text{small } \frac{\gamma}{\omega_0} \qquad (\omega_\gamma \equiv \omega_0\sqrt{1 - \tfrac{1}{4}\gamma^2/\omega_0^2} \simeq \omega_0)$$

Quality factor

$$\boxed{Q \equiv \frac{E(t)}{E(t - \pi/\omega_0) - E(t + \pi/\omega_0)}} \simeq \frac{1}{e^{\pi\gamma/\omega_0} - e^{-\pi\gamma/\omega_0}} = \tfrac{1}{2}\operatorname{csc} h(\pi\gamma/\omega_0)$$

$$\simeq \frac{\omega_0}{2\pi\gamma} \quad (\text{for small } \gamma/\omega_0).$$

Q is the inverse of the fraction of the oscillator’s energy that is dissipated in one period
- approximately the number of oscillations before the energy decays by factor e

SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

- more than 1 unknown function: $y_1(x), y_2(x), \ldots, y_n(x)$
- set of ODEs that couple $y_1, \ldots, y_n$

▷ physical applications: systems with more than 1 degree of freedom. dynamics couples differential equations for different variables.

Example. System of first-order differential equations:

$$y_1' = F_1(x, y_1, y_2, \ldots, y_n)$$

$$y_2' = F_2(x, y_1, y_2, \ldots, y_n)$$

$$\cdots$$

$$y_n' = F_n(x, y_1, y_2, \ldots, y_n)$$

An nth-order differential equation

$$y^{(n)} = G(x, y, y', y'', \ldots, y^{(n-1)})$$

can be thought of as a system of n first-order equations.

- Set new variables $y_1 = y$; $y_2 = y'$; $\ldots$; $y_n = y^{(n-1)}$
  - Then the system of first-order equations

$$y_1' = y_2$$

$$\ldots$$

$$y_{n-1}' = y_n$$

$$y_n' = G(x, y_1, y_2, \ldots, y_n)$$

is equivalent to the starting nth-order equation.

$\diamondsuit$ Systems of linear ODEs with constant coefficients can be solved by a generalization of the method seen for single ODE:

General solution $= \mathrm{PI} + \mathrm{CF}$

$\triangleright$ Complementary function CF by solving
system of auxiliary equations

$\triangleright$ Particular integral PI from a
set of trial functions
with functional form as the inhomogeneous terms

$\diamondsuit$ Warm-up exercise

The variables $\psi(z)$ and $\phi(z)$ obey the simultaneous differential equations

$$3\ \frac{d\phi}{dz} + 5\psi = 2z$$

$$3\ \frac{d\psi}{dz} + 5\phi = 0.$$

Find the general solution for ψ.

$\diamondsuit$ Next time we will consider explicit examples of solution of systems of ODE's with constant coefficients