Second order linear equation with constant coefficients

The particular integral:

\[ Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = h(x) \]
Solutions with combinations of driving functions

\[ Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = h_1(x) + h_2(x) \]

\[ Lf_1 = a_2 \frac{d^2 f_1}{dx^2} + a_1 \frac{df_1}{dx} + a_0 f_1 = h_1(x) \quad Lf_2 = a_2 \frac{d^2 f_2}{dx^2} + a_1 \frac{df_2}{dx} + a_0 f_2 = h_2(x) \]

Since the equation is linear a solution to the original equation is given by

\[ f = f_1 + f_2 \]
Sinusoidal $h$

\[ Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = h(x). \]

\[ h = H \cos x \quad \text{so} \quad Lf \equiv a_2 f'' + a_1 f' + a_0 f = H \cos x \]

\[ Lg(x) \equiv a_2 g'' + a_1 g' + a_0 g = He^{ix} \]

\[ \Re(Lg) = L[\Re(g)] = \Re(He^{ix}) = H\Re(e^{ix}) = H \cos x \]

\[ f = \Re(g) \quad \text{is the solution to the real equation} \]

Solution: \[ g = Pe^{ix} \quad Lg = (-a_2 + ia_1 + a_0)Pe^{ix} \quad \Rightarrow \quad P = \frac{H}{-a_2 + ia_1 + a_0}. \]

\[ f = \Re \left( \frac{He^{ix}}{a_0 - a_2} \right) \]
\[ = H \frac{(a_0 - a_2) \cos x + a_1 \sin x}{(a_0 - a_2)^2 + a_1^2}. \]
Ex 5  \[ f'' + 3f' + 2f = \cos x. \]

\[ g'' + 3g' + 2g = e^{ix}. \]

\[ g = Pe^{ix} \quad \text{where} \quad P = \frac{1}{-1 + 3i + 2}. \]

\[ f_1 = \Re(e^{ix}) = \frac{1}{10} (\cos x + 3\sin x). \]

**CF**  \[ f_0 = Ae^{-x} + Be^{-2x} \]

**General solution**  \[ f = Ae^{-x} + Be^{-2x} + \frac{1}{10} (\cos x + 3\sin x) \]
\[ f'' + 3f' + 2f = \cos x. \]

\[ f = Ae^{-x} + Be^{-2x} + \frac{1}{10}(\cos x + 3\sin x) \]

\[ x(0) = 4, \quad x'(0) = 1 \quad \Longleftrightarrow \quad A = \frac{17}{2}, \quad B = -\frac{23}{5} \]
Ex 5

\[ f'' + 3f' + 2f = \cos x. \]

What if \( \cos(x) \to \sin(x) \)?

\[ g'' + 3g' + 2g = e^{ix}. \]

\[ f = \text{Im}(g) \]

\[ g = Pe^{ix} \quad \text{where} \quad P = \frac{1}{-1 + 3i + 2}. \]

PI

\[ f_1 = \text{Im}(\frac{e^{ix}}{1 + 3i}) = \frac{1}{10} (\sin x - 3 \cos x). \]

CF

\[ f_0 = Ae^{-mx} + Be^{-2mx} \]

General solution

\[ f = Ae^{-mx} + Be^{-2mx} + \frac{1}{10} (\sin x - 3 \cos x) \]
Ex 6 \[ f'' + 3f' + 2f = 3 \cos x + 4 \sin x. \]

\[ = 5 \cos(x + \phi) = 5 \Re(e^{i(x+\phi)}) \]

where \( \phi = \arctan(-4/3) \)

Proof:

\[ \cos(x + \phi) = \cos x \cos \phi - \sin x \sin \phi \]

\[ A \cos x + B \sin x = \sqrt{A^2 + B^2} \left( \frac{A}{\sqrt{A^2 + B^2}} \cos x + \frac{B}{\sqrt{A^2 + B^2}} \sin x \right) \]

\[ = \sqrt{A^2 + B^2} \cos(x + \phi), \]

\[ \cos \phi = A/\sqrt{A^2 + B^2}, \quad \sin \phi = -B/\sqrt{A^2 + B^2} \]

and

\[ \tan \phi = -B/A \]
Ex 6 \[ f'' + 3f' + 2f = 3\cos x + 4\sin x. \]

\[ = 5\cos(x + \phi) = 5\Re(e^{i(x+\phi)}) \]

where \( \phi = \arctan(-4/3) \)

\[ g'' + 3g' + 2g = 5e^{i(x+\phi)} \]

Trial solution: \( g = Pe^{i(x+\phi)} \)

\[ P = \frac{5}{-1 + 3i + 2} = \frac{5}{1 + 3i}, \]

\[ f_1 = 5\Re\left(\frac{e^{i(x+\phi)}}{1 + 3i}\right) = \frac{1}{2}[\cos(x + \phi) + 3\sin(x + \phi)]. \]
Ex 7

\[ f'' + f = \cos x \quad \Rightarrow \quad g'' + g = e^{ix} \]

C.F. \((\frac{d}{dx} + i)(\frac{d}{dx} - i)g = e^{ix} \quad \Rightarrow \quad g = Ce^{ix}, \quad f = A\cos(x) + B\sin(x)\)

P.I. \((\frac{d}{dx} + i)(\frac{d}{dx} - i)g = e^{ix}\).

Try \quad g = Pxe^{ix}

Then \quad e^{ix} = (\frac{d}{dx} + i)(\frac{d}{dx} - i)Pxe^{ix} = (\frac{d}{dx} + i)Pe^{ix} = 2iPe^{ix}

\[ \Rightarrow \quad P = \frac{1}{2i} \quad \Rightarrow \quad f = \Re\left(\frac{xe^{ix}}{2i}\right) = \frac{1}{2}x\sin x \]
Ex 8

\[ f'' + f = e^{-x}(3 \cos x + 4 \sin x) = 5 \Re(e^{-x}e^{i(x+\phi)}) \]

where \( \phi = \arctan(-4/3) \)

Trial function

\[ g = Pe^{(i-1)x+i\phi} \]

\[ P = \frac{5}{(i-1)^2 + 1} = \frac{5}{1-2i}. \]

PI

\[ f_1 = 5 \Re\left(\frac{e^{(i-1)x+i\phi}}{1-2i}\right) = e^{-x}\cos(x+\phi) - 2\sin(x+\phi). \]
Recap

2nd-order linear ODEs with constant coefficients:
\[ a_2 f'' + a_1 f' + a_0 f = h(x) \]

- General solution  =  \( \text{PI} + \text{CF} \)
- \( \text{CF} = c_1 u_1 + c_2 u_2, \)
  \( u_1 \) and \( u_2 \) linearly independent solutions of the homogeneous equation

▷ Complementary function \( \text{CF} \) by solving auxiliary equation
▷ Particular integral \( \text{PI} \) by trial function with functional form of the inhomogeneous term

♠ Next: physical application to forced, damped oscillator
Oscillators

\[ m \ddot{x} = -m \omega_0^2 x - m \gamma \dot{x} + m F \cos \omega t. \]

spring \hspace{1cm} friction \hspace{1cm} forcing

The associated complex equation is

\[ \ddot{z} + \gamma \dot{z} + \omega_0^2 z = F e^{i \omega t}. \]

Transients: \hspace{1cm} CF - Auxiliary equation: \hspace{1cm} \[ z = e^{\alpha t}. \]

\[ \alpha^2 + \gamma \alpha + \omega_0^2 = 0 \quad \Rightarrow \quad \alpha = -\frac{1}{2} \gamma \pm i \sqrt{\omega_0^2 - \frac{1}{4} \gamma^2} \]

\[ = -\frac{1}{2} \gamma \pm i \omega_{\gamma} \quad \text{where} \quad \omega_{\gamma} = \omega_0 \sqrt{1 - \frac{1}{4} \gamma^2 / \omega_0^2}. \]

Complementary function

\[ x = e^{-\gamma t/2} [A \cos(\omega_{\gamma} t) + B \sin(\omega_{\gamma} t)] = e^{-\gamma t/2} N \cos(\omega_{\gamma} t + \psi) \]

Since \( \gamma > 0 \), the CF \( \rightarrow 0 \) as \( t \rightarrow \infty \) \hspace{1cm} .... CF describes "transients"
Steady state solutions

\[ \ddot{z} + \gamma \dot{z} + \omega_0^2 z = F e^{i\omega t}. \]

Particular integral

\[ x = \Re\left( \frac{F e^{i\omega t}}{\omega_0^2 - \omega^2 + i\omega\gamma} \right). \]

i.e. the Particular integral describes the steady state solution after the transients have died away.

Since the denominator \( = \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} \) \( e^{i\phi} \) where \( \phi \equiv \arctan\left( \frac{\omega\gamma}{\omega_0^2 - \omega^2} \right) \) the particular integral can be written as

\[ x = \frac{F \Re(e^{i(\omega t - \phi)})}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} = \frac{F \cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \]

For \( \phi > 0 \), \( x \) achieves the same phase as \( F \) at \( t \) greater by \( \left( \frac{\phi}{\omega} \right) \)

- \( \phi \) is called the "phase lag" of the response.
The amplitude of the response is \( A = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \),

This has a maximum when \( 0 = \frac{dA^2}{d\omega} = -4(\omega_0^2 - \omega^2)\omega + 2\omega\gamma^2 \Rightarrow \omega^2 = \omega_0^2 - \frac{1}{2} \gamma^2. \)

\( \omega_R = \sqrt{\omega_0^2 - \gamma^2/2} \) is called the “resonant” frequency

The frictional coefficient causes the resonant frequency to be less than the normal frequency

\[ \phi \equiv \arctan\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right) \]
Oscillators

\[ m\ddot{x} = -m\omega_0^2 x - m\gamma \dot{x} + mF \cos \omega t. \]

\text{spring} \quad \text{friction} \quad \text{forcing}

\[ F = mF \cos \omega t, \quad x = \frac{F \Re(e^{i(\omega t - \phi)})}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} = \frac{F \cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \]

Power Input

(steady state) \quad P = \frac{\partial W}{\partial t} = F\dot{x} \quad W = \int_{x(t_0)}^{x(t)} F \, dx'

\[ P = F\dot{x} = mF \cos \omega t \times \frac{-F \omega \sin(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \]

\[ = \frac{\omega m F^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} [-\cos(\omega t) \sin(\omega t - \phi)] \]

\[ = -\frac{1}{2} \frac{\omega m F^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \sin(2\omega t - \phi) + \sin(-\phi). \]

Average over a period

\[ \overline{P} = \frac{\frac{1}{2} \omega m F^2 \sin \phi}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}. \]
Energy dissipated

\[ m \ddot{x} = - m \omega_0^2 x - m \gamma \dot{x} + m F \cos \omega t. \]

spring friction forcing

\[
\overline{D} = m \gamma \dot{x} \ddot{x} = \frac{m \gamma \omega^2 F^2 \frac{1}{2}}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}.
\]

\[
x = \frac{F \cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}
\]

\[
\sin^2(\omega t - \phi) = \frac{1}{2}
\]

\[
\overline{D} = \overline{P} = \frac{\frac{1}{2} \omega m F^2 \sin \phi}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}
\]

since

\[
\sin \phi = \gamma \omega \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}
\]

\[
(\phi \equiv \arctan(\frac{\omega \gamma}{\omega_0^2 - \omega^2}))
\]
**Quality Factor**

\[ m\ddot{x} = -m\omega_0^2 x - m\gamma \dot{x} + mF \cos \omega t. \]

- spring
- friction
- forcing

Energy content of transient motion that the CF describes

\[ E = \frac{1}{2} (m\dot{x}^2 + m\omega_0^2 x^2) \]

\[ x = e^{-\gamma t/2} A \cos(\omega \gamma t + \psi) \]

\[ = \frac{1}{2} m A^2 e^{-\gamma t} [\frac{1}{4} \gamma^2 \cos^2 \eta + \omega_\gamma \gamma \cos \eta \sin \eta + \omega_\gamma^2 \sin^2 \eta + \omega_0^2 \cos^2 \eta] \quad (\eta \equiv \omega \gamma t + \psi) \]

\[ E \approx \frac{1}{2} m (\omega_0 A)^2 e^{-\gamma t} \quad \text{small } \frac{\gamma}{\omega_0} \quad (\omega_\gamma \equiv \omega_0 \sqrt{1 - \frac{1}{4} \gamma^2/\omega_0^2} = \omega_0) \]

**Quality factor**

\[ Q \equiv \frac{E(t)}{E(t - \pi/\omega_0) - E(t + \pi/\omega_0)} \approx \frac{1}{e^{\pi \gamma/\omega_0} - e^{-\pi \gamma/\omega_0}} = \frac{1}{2} \csc h(\pi \gamma/\omega_0) \]

\[ \approx \frac{\omega_0}{2\pi \gamma} \quad \text{(for small } \gamma/\omega_0). \]

Q is the inverse of the fraction of the oscillator’s energy that is dissipated in one period - approximately the number of oscillations before the energy decays by factor e.
SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

• more than 1 unknown function: \( y_1(x), y_2(x), \ldots, y_n(x) \)

• set of ODEs that couple \( y_1, \ldots, y_n \)

▷ physical applications: systems with more than 1 degree of freedom. dynamics couples differential equations for different variables.

Example. System of first-order differential equations:

\[
y'_1 = F_1(x, y_1, y_2, \ldots, y_n)
\]

\[
y'_2 = F_2(x, y_1, y_2, \ldots, y_n)
\]

\[
\ldots
\]

\[
y'_n = F_n(x, y_1, y_2, \ldots, y_n)
\]
An $n$th-order differential equation

$$y^{(n)} = G(x, y, y', y'', \ldots, y^{(n-1)})$$

can be thought of as a system of $n$ first-order equations.

- Set new variables $y_1 = y; y_2 = y'; \ldots; y_n = y^{(n-1)}$
- Then the system of first-order equations

$$y_1' = y_2$$

$$\vdots$$

$$y_{n-1}' = y_n$$

$$y_n' = G(x, y_1, y_2, \ldots, y_n)$$

is equivalent to the starting $n$th-order equation.
Systems of linear ODEs with constant coefficients can be solved by a generalization of the method seen for single ODE:

$\text{General solution} = \text{PI} + \text{CF}$

$\implies$ Complementary function CF by solving system of auxiliary equations

$\implies$ Particular integral PI from a set of trial functions with functional form as the inhomogeneous terms
Warm-up exercise

The variables $\psi(z)$ and $\phi(z)$ obey the simultaneous differential equations

$$3 \frac{d\phi}{dz} + 5\psi = 2z$$
$$3 \frac{d\psi}{dz} + 5\phi = 0.$$ 

Find the general solution for $\psi$.

Next time we will consider explicit examples of solution of systems of ODE’s with constant coefficients.