

SECOND-ORDER LINEAR ODEs

$$f'' + p(x)f' + q(x)f = h(x)$$

◇ Generalities

◇ Structure of general solution

◇ Equations with constant coefficients

Second order linear equations

General form :

$$\frac{d^2 f}{dx^2} + p(x) \frac{df}{dx} + q(x)f = h(x).$$

Integrating factor? Suppose $\exists I(x)$ such that $\frac{d^2 If}{dx^2} = Ih$

 $2 \frac{dI}{dx} = Ip \quad \text{and} \quad \frac{d^2 I}{dx^2} = Iq.$

These equations are incompatible in most cases....

Structure of the general solution (GS)

$$f'' + p(x)f' + q(x)f = h(x)$$

♠ The general solution f is the sum of a particular solution f_0 (the “particular integral”, PI) and the general solution f_1 of the associated homogeneous equation (the “complementary function”, CF):

$$f = f_0 + f_1 ,$$

$$\text{i.e., GS} = \text{PI} + \text{CF} .$$

♠ The complementary function CF is given by linear combination of two *linearly independent* ($\hookrightarrow$ see next) solutions u_1 and u_2 :

$$\text{CF} = c_1 u_1(x) + c_2 u_2(x)$$

c_1 and c_2 arbitrary constants

♠ Two functions $u_1(x)$ and $u_2(x)$ are *linearly independent* if the relation $\alpha u_1(x) + \beta u_2(x) = 0$ implies $\alpha = \beta = 0$.

$$\text{Let } \alpha u_1(x) + \beta u_2(x) = 0.$$

$$\text{Differentiating } \Rightarrow \alpha u_1'(x) + \beta u_2'(x) = 0.$$

$$\bullet \text{ If } W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_2 u_1' \neq 0$$

then $\alpha = \beta = 0$, and u_1 and u_2 are linearly independent.

$$\bullet \text{ If } W(u_1, u_2) = u_1 u_2' - u_2 u_1' = 0$$

then $u_2 = \text{constant} \times u_1 \Rightarrow u_1$ and u_2 not linearly independent

$W(u_1, u_2)$ = wronskian determinant of functions u_1 and u_2

Example: The functions $u_1(x) = \sin x$ and $u_2(x) = \cos x$ are linearly independent.

$$\text{Let } \alpha \sin x + \beta \cos x = 0 .$$

$$\text{Differentiating } \Rightarrow \alpha \cos x - \beta \sin x = 0 .$$

$$\text{So } \alpha = \beta \frac{\sin x}{\cos x} \Rightarrow \beta \left(\frac{\sin^2 x}{\cos x} + \cos x \right) = 0 \Rightarrow \beta \frac{1}{\cos x} = 0 \Rightarrow \beta = 0 .$$

$$\text{Thus } \alpha = \beta = 0 .$$

• Alternatively:

$$W(u_1, u_2) = u_1 u_2' - u_2 u_1' = -\sin^2 x - \cos^2 x = -1 \Rightarrow \text{linear independence}$$

Homogeneous equation:

$$f'' + p(x)f' + q(x)f = 0$$

♠ Any solution u can be written as a linear combination of linearly independent solutions u_1 and u_2 .

Since u , u_1 and u_2 all solve the homogeneous eq., with nonzero coefficients of the second-derivative, first-derivative and no-derivative terms, we must have $\det = 0$

$$\begin{vmatrix} u & u_1 & u_2 \\ u' & u_1' & u_2' \\ u'' & u_1'' & u_2'' \end{vmatrix} = 0$$

$W(u, u_1, u_2) = 0$ for solutions u , u_1 , u_2

$\Rightarrow \alpha u + \beta u_1 + \gamma u_2 = 0$ for α , β and γ not all zero.

Solving for u expresses the solution u as a linear combination of u_1 and u_2 .

$$\Rightarrow \text{CF} = c_1 u_1(x) + c_2 u_2(x)$$

• solutions span whole set of linear combinations of two independent u_1 , u_2

Example: The general solution of the 2nd-order linear ODE

$$y'' + y = 0$$

is $A \sin x + B \cos x$.

(simple harmonic oscillator)

To show this, it is sufficient to show that

- i) $\sin x$ and $\cos x$ solve the equation (e.g. by direct computation) and
 - ii) $\sin x$ and $\cos x$ are linearly independent (see previous Example).
- Then general theorem CF = $c_1 u_1(x) + c_2 u_2(x)$ yields the result.

- Useful reference for this part of the course, with worked problems and examples, is

Schaum's Outline Series

Differential Equations

R. Bronson and G. Costa

McGraw-Hill (Third Edition, 2006)

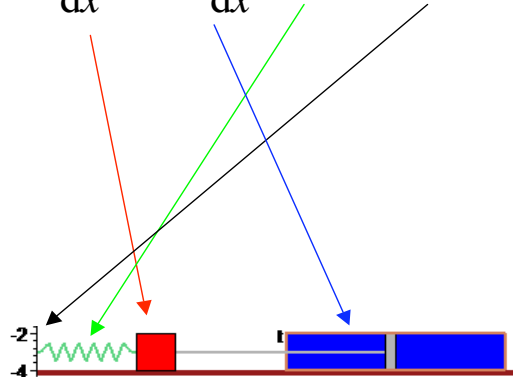
◇ See chapters 8 to 14.

2nd-order linear ODEs with constant coefficients:

- general methods of solution available
- arise in many physical applications

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = h(x).$$



Second order linear equation with constant coefficients

Solution

$$Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = h(x).$$

The number of independent complementary functions is the number of integration constants – equal to the order of the differential equation

Complementary function

1) Construct f_0 the general solution to the homogeneous equation $Lf_0 = 0$

2) Find a solution, f_1 , to the inhomogeneous equation $Lf_1 = h$

Particular integral

General solution : $f_0 + f_1$

For a nth order differential equation need n independent solutions to $Lf=0$ to specify the complementary function

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = h(x).$$

Complementary function

$$Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = 0.$$

Try $y = e^{mx}$  $a_2 m^2 + a_1 m + a_0 = 0.$

$$m_{\pm} \equiv \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2},$$

“Auxiliary” equation

$a_1^2 - 4a_2 a_0 \rightarrow +, 0, -$



Complementary function

$$y = A_+ e^{m_+ x} + A_- e^{m_- x}.$$

Two constants of integration

Ex 1

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 3y = 0.$$

Auxiliary eq. $(m+3)(m+1) = 0 \Rightarrow$ CF is $y = Ae^{-3x} + Be^{-x}$

$$\text{If } y(0)=2, y'(0)=0 \Rightarrow A+B=2, \quad -3A-B=0$$

Initial
conditions

$$A=-1, B=3 \Rightarrow y = -e^{-3x} + 3e^{-x}$$

Ex 2

$$Ly = \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 5y = 0.$$

Auxiliary eq. $m^2 + 2m + 5 = 0$

i.e. $m = \frac{1}{2}(-2 \pm \sqrt{4 - 20}) = -1 \pm 2i \Rightarrow$ CF is $y = Ae^{(-1+2i)x} + Be^{(-1-2i)x}$

Complex?

But L is a real operator $\Rightarrow 0 = \Re(Ly) = L[\Re(y)]$

i.e. $\Re(y)$ is a solution (as is $\Im(y)$) $\Rightarrow \Re(y) = e^{-x}[A' \cos(2x) + B' \sin(2x)].$

Find the solution for which $y(0) = 5$ and $(dy/dx)_0 = 0$

Initial conditions

$\Rightarrow$ $5 = A'$
 $0 = -A' + 2B' \Rightarrow B' = \frac{5}{2} \Rightarrow y = e^{-x}[5 \cos(2x) + \frac{5}{2} \sin(2x)].$

Factorisation of operators

We wish to solve $Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = 0$.

We did this by trying $y = e^{mx}$

$$\Rightarrow a_2 m^2 + a_1 m + a_0 = a_2 (m - m_+)(m - m_-) = 0. \quad m_{\pm} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

This is equivalent to factorising the equation

$$\begin{aligned} \left(\frac{d}{dx} - m_-\right)\left(\frac{d}{dx} - m_+\right)f &= \frac{d^2 f}{dx^2} - (m_- + m_+) \frac{df}{dx} + m_- m_+ f \\ &= \frac{d^2 f}{dx^2} + \frac{a_1}{a_2} \frac{df}{dx} + \frac{a_0}{a_2} \equiv \frac{Lf}{a_2} \end{aligned}$$

Now we can see why the CF is made up of exponentials because :

$$\left(\frac{d}{dx} - m_-\right)e^{m_- x} = 0 \quad ; \quad \left(\frac{d}{dx} - m_+\right)e^{m_+ x} = 0.$$

Factorisation of operators and repeated roots

$$Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = 0.$$

$$\left(\frac{d}{dx} - m_-\right)\left(\frac{d}{dx} - m_+\right)f = 0 \quad m_{\pm} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

What happens if $a_1^2 - 4a_2a_0 = 0$ and $m_+ = m_- = m$?

$$Lf = \left(\frac{d}{dx} - m\right)\left(\frac{d}{dx} - m\right)f.$$

e^{mx} gives one solution : $L(e^{mx}) = \left(\frac{d}{dx} - m\right)\left(\frac{d}{dx} - m\right)e^{mx} = 0$

xe^{mx} gives the second independent solution :

$$L(xe^{mx}) = \left(\frac{d}{dx} - m\right)\left(\frac{d}{dx} - m\right)xe^{mx} = \left(\frac{d}{dx} - m\right)e^{mx} = 0,$$

$$y = Ae^{mx} + Bxe^{mx}$$

Ex 4 $\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = 0.$

Auxiliary equation $(m+1)^2 = 0$

➡ $y = Ae^{-x} + Bxe^{-x}$

Ex 5 $\frac{d^4 y}{dx^4} - 2 \frac{d^3 y}{dx^3} + 2 \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 0.$

Auxiliary equation $(m-1)^2 (m-i)(m+i) = 0$

➡ $y = e^x (A + Bx) + C \cos x + D \sin x.$

Summary

2nd order linear ODEs: $f'' + p(x)f' + q(x)f = h(x)$

- General solution = PI + CF

- CF = $c_1u_1 + c_2u_2$

- u_1 and u_2 linearly independent solutions
of the homogeneous equation

- Equations with constant coefficients:

- ▷ Solve auxiliary equation to find complementary function CF

- ◇ distinct real roots

- ◇ repeated roots

- ◇ complex roots

- ▷ Next: methods to find the particular integral PI