FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

\[ G(x, y, y') = 0 \]

\( \diamond \) in normal form:

\[ y' = F(x, y) \]

\( \diamond \) in differential form:

\[ M(x, y)dx + N(x, y)dy = 0 \]

- Last time we discussed first-order linear ODE: \( y' + q(x)y = h(x) \).
  We next consider first-order nonlinear equations.
NONLINEAR FIRST-ORDER ODEs

• No general method of solution for 1st-order ODEs beyond linear case; rather, a variety of techniques that work on a case-by-case basis.

Examples:

i) Bring equation to separated-variables form, that is, \[ y' = \frac{\alpha(x)}{\beta(y)}; \]
then equation can be integrated.
Cases covered by this include \[ y' = \varphi(ax + by); \quad y' = \varphi(y/x). \]

ii) Reduce to linear equation by transformation of variables.
Examples of this include Bernoulli’s equation.

iii) Bring equation to exact-differential form, that is
\[ M(x, y)dx + N(x, y)dy = 0 \] such that \[ M = \frac{\partial \phi}{\partial x}, \quad N = \frac{\partial \phi}{\partial y}. \]
Then solution determined from \[ \phi(x, y) = \text{const}. \]
Useful reference for the ODE part of this course
(worked problems and examples)

Schaum’s Outline Series
Differential Equations
R. Bronson and G. Costa

 Chapters 1 to 7: First-order ODE.
First order nonlinear equations

Although no general method for solution is available, there are several cases of physically relevant nonlinear equations which can be solved analytically:

Separable equations

\[
dy = \frac{f(x)}{g(y)}
dx
\]

Solution:

\[\int g(y)dy = \int f(x)dx\]

Ex 1

\[
\frac{dy}{dx} = y^2e^x
\]

\[\Rightarrow\]

\[\int \frac{dy}{y^2} = \int e^x dx\]

i.e \[\frac{-1}{y} = e^x + c\]  or \[y = \frac{-1}{(e^x + c)}\]
Almost separable equations

\[
\frac{dy}{dx} = f(ax + by)
\]

Change variables: \( z = ax + by \)

\[
\frac{dz}{dx} = a + bf(z) \quad \Rightarrow \quad x = \int \frac{1}{(a + bf(z))} \, dz.
\]

Ex 2

\[
\frac{dy}{dx} = (-4x + y)^2
\]

\[
z = y - 4x \quad \Rightarrow \quad \frac{dz}{dx} = -4 + \frac{dy}{dx} = z^2 - 4
\]

\[
x = \frac{1}{4} \ln\left(\frac{z-2}{z+2}\right) + C
\]

\[
\Rightarrow \quad y = 4x + 2 \left(\frac{1+ke^{4x}}{1-ke^{4x}}\right) \quad \text{k a constant}
\]
Homogeneous equations

\[ \frac{dy}{dx} = f(y/x). \]

The equation is invariant under \( x \to sx, \quad y \to sy \quad \text{... homogeneous} \)

Solution

\[ y = vx \quad \Rightarrow \quad y' = v'x + v. \]

\[ \text{i.e.} \quad v' = \frac{1}{x}(f(v) - v) \]

\[ \int \frac{dv}{f(v) - v} = \int \frac{dx}{x} = \ln x + \text{constant}. \]
Ex 3 \[ xy \frac{dy}{dx} - y^2 = (x + y)^2 e^{-y/x} \] Homogeneous

Change variables \( y = vx \quad \Rightarrow \quad y' = v'x + v. \)

\[
(v'x + v) - v = \frac{(1 + v)^2}{v} e^{-v} \quad \Rightarrow \quad \ln x = \int \frac{e^v v dv}{(1 + v)^2}.
\]

To evaluate integral change variables \( u \equiv 1 + v \)

\[
e^{-1} \int \left( \frac{1}{u} - \frac{1}{u^2} \right) e^u du = e^{-1} \left[ \frac{e^u}{u} \right].
\]

\[ i.e. \quad \ln x = \frac{e^x}{1 + \frac{y}{x}} \]
Homogeneous but for constants

\[ \frac{dy}{dx} = \frac{x + 2y + 1}{x + y + 2} \]

\[ x = x' + a, \quad y = y' + b \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy'}{dx'} = \frac{dy'}{dx'} \frac{dx'}{dx} = \frac{dy'}{dx'} \]

\[ \frac{dy'}{dx'} = \frac{x' + 2y' + 1 + a + 2b}{x' + y' + 2 + a + b} \]

\[ 1 + a + 2b = 0 \]
\[ a = -3, \quad b = 1 \]
\[ 2 + a + b = 0 \]

\[ \frac{dy'}{dx'} = \frac{x' + 2y'}{x' + y'} \]

Homogeneous
The Bernoulli equation

\[ \frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \neq 1 \]

To solve, change variable to

\[ z = y^{1-n} \quad \Rightarrow \quad \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx} \]

Gives the equation

\[ \frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x) \]

1st order Linear

Ex 4

\[ y' + y = y^{2/3} \]

\[ z = y^{1-n} = y^{1/3} \quad \Rightarrow \quad z' + \frac{z}{3} = \frac{1}{3} \]

1st order linear

Integrating factor \( e^{x/3} \quad \Rightarrow \quad ze^{x/3} = \int e^{x/3} dx / 3 \)

\[ z = y^{1/3} = 1 + ce^{-x/3} \]
Exercise:
Solve the equation \( 2 \, y' = y/x + x^2/y \)
with initial condition \( y(1) = 2 \).

- This equation is Bernoulli with \( n = -1 \).
  - Set \( z = y^2 \). Then \( z' - z/x = x^2 \).
  - Integrating factor \( I(x) = 1/x \)

\[
\Rightarrow z(x) = x \left[ \int \frac{dx}{x} \frac{x^2}{x} + \text{const.} \right] = \frac{x^3}{2} + \text{const.} \, x
\]

Thus \( y = z^{1/2} = \pm \sqrt{\frac{x^3}{2} + \text{const.} \, x} \)

- Initial condition \( y(1) = 2 \)

\[
\Rightarrow y(x) = \sqrt{\frac{x^3 + 7x}{2}}
\]
Exact equations

• A first-order ODE

\[ M(x, y) \, dx + N(x, y) \, dy = 0 \]

is exact if there exists a function \( \phi(x, y) \) such that

\[ \frac{\partial \phi}{\partial x} = M , \quad \frac{\partial \phi}{\partial y} = N . \]

• In this case the differential equation can be recast as

\[ d\phi = M(x, y) \, dx + N(x, y) \, dy = 0 \]

so that the solution to it is determined by

\[ \phi(x, y) = \text{constant} . \]
Example: Solve the equation \( xy' = -2 \tan y \).

- This equation can be rewritten as
  \[
  2x \sin y \, dx + x^2 \cos y \, dy = 0 ,
  \]
  i.e., \( M(x, y) = 2x \sin y \), \( N(x, y) = x^2 \cos y \),

  which is exact because
  \[
  \frac{\partial \phi}{\partial x} = 2x \sin y \quad \Rightarrow \quad \phi(x, y) = x^2 \sin y + \alpha(y)
  \]
  \[
  \frac{\partial \phi}{\partial y} = x^2 \cos y \quad \Rightarrow \quad x^2 \cos y + \alpha'(y) = x^2 \cos y \quad \Rightarrow \quad \alpha = \text{constant}
  \]

- Therefore \( \phi(x, y) = x^2 \sin y + c \),
  and the general solution is determined by \( x^2 \sin y = \text{const.} \):
  \[
  \Rightarrow \quad y(x) = \arcsin\left(\frac{\text{const.}}{x^2}\right)
  \]
DIFFERENTIAL EQUATIONS AND FAMILIES OF CURVES

- General solution of a first-order ODE \( y' = f(x, y) \) contains an arbitrary constant: \( y = (x, c) \)
  - one curve in \( x, y \) plane for each value of \( c \)
  - general solution can be thought of as one-parameter family of curves

Example: \( y' = -\frac{x}{y} \).

separable equation \( \Rightarrow \int y \, dy = -\int x \, dx \Rightarrow y^2/2 = -x^2/2 + c \)

i.e., \( x^2 + y^2 = \) constant: family of circles centered at origin

![Fig.1](image-url)
Orthogonal trajectories

• Given the family of curves representing solutions of ODE \( y' = f(x, y) \), orthogonal trajectories are given by a second family of curves which are solutions of

\[
y' = -1/f(x, y).
\]

◊ Then each curve in either family is perpendicular to every curve in the other family.

Example:
Find the orthogonal trajectories to the family of circles \( y' = -x/y \).

• Solve \( y' = y/x \).

\[
\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln y = \ln x + \text{constant}
\]

i.e., \( y = cx \) : family of straight lines through the origin
a) Find the family of curves corresponding to solutions of the ODE
\[ y' = \frac{(y^2 - x^2)}{(2xy)}. \]
b) Find the orthogonal trajectories to the above family of curves.

- **homogeneous equation** \( y' = f(y/x) \) with \( f(y/x) = \frac{(y/x - x/y)}{2} \)
solvable by \( y \to v = y/x \) and separation of variables
  \[ x^2 + y^2 = cx : \text{ family of circles tangent to } y - \text{axis at } 0 \]

- **orthogonal trajectories** found by solving \( y' = -\frac{2xy}{(y^2 - x^2)} \)
  \[ x^2 + y^2 = ky : \text{ family of circles tangent to } x - \text{axis at } 0 \]
EXPLOITING FIRST-ORDER METHODS TO TREAT EQUATIONS OF HIGHER ORDER IN SPECIAL CASES

♣ \( y \) not present in 2nd-order equation \( F(x, y, y', y'') = 0 \)
\[ \Rightarrow \text{setting } y' = q \text{ yields 1st-order equation for } q(x). \]

♣ \( x \) not present in 2nd-order equation \( F(x, y, y', y'') = 0 \)
\[ \Rightarrow \text{setting } y' = q, y'' = dq/dx = q(dq/dy) \text{ yields } G(y, q, dq/dy) = 0. \]

Example: homogeneous, flexible chain hanging under its own weight

Using Newton’s law, the shape \( y(x) \) of the chain obeys the 2nd–order nonlinear differential equation
\[ y'' = a \sqrt{1 + (y')^2}, \quad a = \rho \ g / T \]

Setting \( y' = q \quad \Rightarrow \quad q' = a \sqrt{1 + q^2} \)
Separation of variables ⇒ \[ \int \frac{1}{\sqrt{1 + q^2}} \, dq = a \int \, dx \]

Using \( q = \frac{dy}{dx} = 0 \) at \( x = 0 \) ⇒ \( \ln(q + \sqrt{1 + q^2}) = ax \)

Solving for \( q \) ⇒ \( q = \frac{dy}{dx} = \frac{e^{ax} - e^{-ax}}{2} \)

Thus \( y(x) = \frac{1}{a} \frac{e^{ax} + e^{-ax}}{2} + \text{constant} = \frac{1}{a} \cosh(ax) + \text{constant} \)

This curve is called a catenary.

Historical note. The problem of the catenary was the subject of a challenge posed by Jakob Bernoulli in 1690, in response to which the problem was solved the following year independently by Johann Bernoulli, Leibniz and Huygens.
Summary

◊ No general method of solution for 1st-order ODEs beyond linear case; rather, a variety of techniques that work on a case-by-case basis.

Main guiding criteria:
- methods to bring equation to separated-variables form
- methods to bring equation to exact differential form
- transformations that linearize the equation

◊ 1st-order ODEs correspond to families of curves in $x, y$ plane
  $\Rightarrow$ geometric interpretation of solutions

◊ Equations of higher order may be reducible to first-order problems in special cases — e.g. when $y$ or $x$ variables are missing from 2nd order equations