INTRODUCTION TO DIFFERENTIAL EQUATIONS

- Equations involving an unknown function and its derivatives

$$\text{ex.}: \quad \frac{df}{dx} + 2xf = e^{-x^2}$$

▷ solution for f specified by equation + initial data [e.g., value of f at a point]

- Physical laws encoded in differential equations

• In this course we will talk of

ordinary differential equations:

involving derivatives with respect to one variable

• to be contrasted with

partial differential equations:

involving derivatives with respect to more than one variable

◊ Partial DE will occur in Hilary Term course

"Normal modes, wave motion and the wave equation"

B. Ordinary differential equations (ODE)

I. Generalities

Differential operators.

Initial Conditions.

Linearity.

General solutions and particular integrals.

II. First order ODEs

The linear case.

Non-linear equations.

III. Second order linear ODEs

General structure of solutions.

Equations with constant coefficients.

Application: the mathematics of oscillations.

IV. Systems of first-order differential equations

Solution methods for linear systems.

Application: electrical circuits.

Differential Equations

e.g. $\frac{\mathrm{d}f}{\mathrm{d}x} + xf = \sin x$ + Initial conditions

Differential operators

Functions : map numbers ⟶ numbers $x \rightarrow e^{x}$

Operators : map functions ⟶ functions $f \rightarrow \alpha f ; f \rightarrow 1/f ; f \rightarrow f + \alpha$

Differential operators : $f \rightarrow \frac{\mathrm{d}f}{\mathrm{d}x} ; f \rightarrow \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} ; f \rightarrow 2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\frac{\mathrm{d}f}{\mathrm{d}x} ; ...)$

Convenient to name operator *e.g.* $L(f) \;:\; f \rightarrow \frac{\mathrm{d}f}{\mathrm{d}x}$

Order of a differential operator

$$L_1(f) \equiv \frac{\mathrm{d}f}{\mathrm{d}x} + 3f \quad \text{is first order,}$$

$$L_2(f) \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 3f \quad \text{is second order,}$$

$$L_3(f) \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 4\frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{is second order.}$$

Linear operator

α, β real or complex numbers

$$\text{If } L(\alpha f + \beta g) = \alpha L(f) + \beta L(g),$$

then L is a linear operator

e.g. $f \rightarrow \frac{\mathrm{d}f}{\mathrm{d}x}$ and $f \rightarrow \alpha f$ are linear

$f \rightarrow \frac{1}{f}$ and $f \rightarrow f + \alpha$ are not linear

The principle of superposition

Suppose f and g are solutions to $L(y)=0$ for different initial conditions

$$i.e. \quad L(f)=0, \quad L(g)=0$$

Consider the Linear Combination : $\alpha f + \beta g$

If L linear then $\mathrm{L}(\alpha f + \beta g) = \alpha L(f) + \beta L(g) = 0$

i.e. a linear combination of solutions of a linear operator is also a solution –

"principle of superposition"

Inhomogeneous terms

$$L(f) = h(x)$$

e.g. $\frac{df}{dx} + xf = \sin x$

Sometimes called "driving" term

$$Lf = 0$$ homogeneous differential equation

ex. : $\frac{df}{dx} + xf = 0$

$$Lf = h(x) \neq 0$$ inhomogeneous differential equation

ex. : $\frac{df}{dx} + xf = \sin x$

Solution to differential equations

$$L(f) = h(x)$$

The number of independent complementary functions is the number of integration constants – equal to the order of the differential equation

Complementary function

1) Construct f_0 the general solution to the homogeneous equation $Lf_0 = 0$

2) Find a solution, f_1, to the inhomogeneous equation $Lf_1 = h$

Particular integral

General solution : $f_0 + f_1$

For a nth order differential equation need n independent solutions to Lf=0 to specify the complementary function

First order linear equations

General form : $\frac{\mathrm{d}f}{\mathrm{d}x} + q(x)f = h(x).$

Easy to solve

Integrating factor

Look for a function I(x) such that $\mathrm{I(x)}\frac{\mathrm{d}f}{\mathrm{d}x} + I(x)q(x)f \equiv \frac{dIf}{dx} = I(x)h(x)$

Solution: CF $\frac{dIf}{dx} = 0$ is $If = const$

PI $\mathrm{I(x)}f(x) = \int^{x} I(x')h(x')dx'$

Solution : $f(x) = \frac{1}{I(x)} \int_{x_0}^{x} I(x')h(x')dx'$

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First order :
1 integration constant (CF)

We have $I(x)q(x) = \frac{dI}{dx}$

$$\Rightarrow \ln(I(x)) = \int^{x} q(x')dx' \quad \Rightarrow \quad I(x) = e^{\int^{x} q(x')dx'}$$

"Integrating factor"

Ex1 Solve

$$2x\frac{\mathrm{d}f}{\mathrm{d}x} - f = x^2.$$

Writing in “standard” form : $\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{f}{2x} = \tfrac{1}{2}x$

$$\text{so } q = -\frac{1}{2x} \text{ and } I = \mathrm{e}^{-\frac{1}{2}\ln x} = \frac{1}{\sqrt{x}}.$$

Plugging this into the form of the solution we have :

$$f = \tfrac{1}{2}\sqrt{x}\int_{x_0}^{x}\sqrt{x'}\,\mathrm{d}x' = \tfrac{1}{3}(x^2 - x_0^{3/2}x^{1/2})$$

Example 2:

Solve $f' + 2f = x^2$

with $f(3) = 0.$

Integrating factor $I(x) = \exp(\int 2\ dx) = \exp(2x)$

$$\Rightarrow\ f(x) = \exp(-2x)[\int dx\ x^2 \exp(2x) + \text{const.}]$$

$$= \exp(-2x)[\exp(2x)(x^2 - x + 1/2)/2 + \text{const.}]$$

Initial condition $f(3) = 0 \ \Rightarrow\ \exp(-6)[\exp(6)13/4 + \text{const.}] = 0$

$$\Rightarrow\ \text{const.} = -\exp(6)13/4$$

Thus $f(x) = (x^2 - x + 1/2)/2 - (13/4)\exp(6 - 2x)$

Example 3:

Solve $f' + \sin x \; f = x$

with $f(0) = 2$.

- Integrating factor $I(x) = \exp(\int \sin x \; dx) = \exp(-\cos x)$

$$\Rightarrow \;\; f(x) = \exp(\cos x)[\int dx \;\, x \exp(-\cos x) + \mathrm{const.}]$$

- Initial condition $f(0) = 2 \;\; \Rightarrow$

$$\Rightarrow \;\; f(x) = \exp(\cos x)[2/e + \int_0^x dt \;\, t \exp(-\cos t)]$$

♢ The discussion of 1st-order linear ODE shows that the general solution depends on one arbitrary constant, and one initial condition is needed to specify the solution uniquely.

- What is the number of initial conditions necessary to specify solution uniquely for a linear ODE of order n

Consider general form $\quad (f^{(k)} = d^k f / dx^k)$

$$Lf = a_n f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f = 0$$

Rearrange this as

$$f^{(n)} = -\frac{a_{n-1}}{a_n} f^{(n-1)} - \ldots - \frac{a_0}{a_n} f$$

Now differentiate both sides wrt x and re-express $f^{(n)}$ in rhs using above equation:

$f^{(n+1)} =$ in terms of f and its derivatives up to $f^{(n-1)}$

Differentiate once more, and by the same reasoning

$$f^{(n+2)} = \text{in terms of } f \text{ and its derivatives up to } f^{(n-1)}$$

So any derivative $f^{(k)}$ can be expressed in terms of f and derivatives up to $f^{(n-1)}$. Now use Taylor series expansion

$$f(x) = f(x_0) + (x - x_0)f^{(1)}(x_0) + \frac{1}{2}(x - x_0)^2 f^{(2)}(x_0) + \dots$$

Any of the derivatives at x_0 is expressed in terms of

$$f(x_0), \dots, f^{(n-1)}(x_0).$$

Therefore f at x is determined by original n-th order DE $+\ n$ conditions giving the n constants $f^{(r)}(x_0)$, $r = 0, \dots, n-1$ (or equivalently an alternative set of n constants).

Summary

- Superposition principle for linear homogeneous ODEs.

  If f_1 and f_2 are solution of $Lf = 0$, then any linear combination $\alpha f_1 + \beta f_2$ is also solution.

- General solution of linear inhomogeneous ODE $Lf = h$ is sum of the particular integral (PI) and complementary function (CF).

- First-order linear ODE solvable by general method:

$$f' + qf = h$$

$$\Rightarrow \; f = e^{-\int q} \left[\int e^{\int q} \, h + c \right] .$$

- n initial conditions needed to specify solution of linear ODE of order n