2nd-order linear ODEs with constant coefficients:

$$a_2 f'' + a_1 f' + a_0 f = h(x)$$

$\triangleright$ We have seen methods to find the complementary function CF by solving the auxiliary equation.

$\triangleright$ Next: methods to find the particular integral PI.

Second order linear equation with constant coefficients

The particular integral :

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x)$$

Exponential h(x)

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x) = H\mathrm{e}^{\gamma x}$$

$$\boxed{\equiv a_2 \left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_2\right) f = H\mathrm{e}^{\gamma x}.}$$

$$\mathrm{CF} = A_{1,2}\mathrm{e}^{\alpha_{1,2}x}.$$

PI : $$\boxed{f = \frac{H\mathrm{e}^{\gamma x}}{a_2(\gamma - \alpha_1)(\gamma - \alpha_2)}}$$

$$\alpha_1 \neq \alpha_2 \neq \gamma \neq \alpha_1$$

$$\boxed{f = \frac{Hx\mathrm{e}^{\alpha_2 x}}{a_2(\alpha_2 - \alpha_1)}}$$

$$\gamma = \alpha_2 \neq \alpha_1$$

$$\mathrm{CF} = A\mathrm{e}^{\alpha x} + Bxe^{\alpha x}$$

$$\alpha_1 = \alpha_2 = \alpha = \gamma$$

PI : $Try \quad f = Px^2\mathrm{e}^{\alpha x}$

$$a_2\left(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha_2\right)f = H\mathrm{e}^{\gamma x}.$$

$$Try \quad f = Px^2\mathrm{e}^{\alpha x}$$

$$a_2\left(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\right)^2 Px^2\mathrm{e}^{\alpha x} = a_2\left(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\right)2Px\mathrm{e}^{\alpha x} = 2a_2P\mathrm{e}^{\alpha x} \quad = H\mathrm{e}^{\alpha x}$$

$$\Rightarrow \quad \boxed{P = \frac{H}{2a_2}}$$

Exponential h(x)

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

$$\mathrm{CF} = A_{1,2} \mathrm{e}^{\alpha_{1,2} x}.$$

PI :

$$\boxed{f = \frac{H\mathrm{e}^{\gamma x}}{a_2(\gamma - \alpha_1)(\gamma - \alpha_2)}}$$

$$\alpha_1 \neq \alpha_2 \neq \gamma \neq \alpha_1$$

$$\boxed{f = \frac{Hx\mathrm{e}^{\alpha_2 x}}{a_2(\alpha_2 - \alpha_1)}}$$

$$\gamma = \alpha_2 \neq \alpha_1$$

$$\boxed{f = \frac{H}{2a_2} x^2 \mathrm{e}^{\alpha x}}$$

$$\alpha_1 = \alpha_2 = \alpha = \gamma$$

Ex 4

$$\boxed{f'' + 3f' + 2f = \mathrm{e}^{-x}, \quad f(0) = 1, \quad f'(0) = 1}$$

The CF is $f_0 = A\mathrm{e}^{-2x} + B\mathrm{e}^{-x}$

PI ... try $f_1 = Px\mathrm{e}^{-x}$

$$\mathrm{e}^{-x} = (\frac{\mathrm{d}}{\mathrm{d}x} + 2)(\frac{\mathrm{d}}{\mathrm{d}x} + 1)Px\mathrm{e}^{-x} = (\frac{\mathrm{d}}{\mathrm{d}x} + 2)P\mathrm{e}^{-x}$$

$$= P\mathrm{e}^{-x}.$$

$$\boxed{P = 1 \quad \text{and} \quad f_1 = x\mathrm{e}^{-x}}$$

Full solution : $f = f_0 + f_1 = x\mathrm{e}^{-x} + A\mathrm{e}^{-2x} + B\mathrm{e}^{-x}$

Initial conditions $\Rightarrow$ $f = x\mathrm{e}^{-x} - \mathrm{e}^{-2x} + 2\mathrm{e}^{-x}$

Solutions with combinations of driving functions

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h_1(x) + h_2(x)$$

$$Lf_1 = a_2 \frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_1}{\mathrm{d}x} + a_0 f_1 = h_1(x) \qquad Lf_2 = a_2 \frac{\mathrm{d}^2 f_2}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_2}{\mathrm{d}x} + a_0 f_2 = h_2(x)$$

Since the equation is linear a solution to the original equation is given by

$$f = f_1 + f_2$$

Homework

(Maths Collection Paper 2009)

- Find the general solution of the following equation:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = e^{-2x} \ .$$

CF: Auxiliary equation $m^2 - m - 6 = 0 \ \Rightarrow \ m_+ = 3 \ , \ m_- = -2.$

So $\mathrm{CF} = Ae^{3x} + Be^{-2x}$.

PI: Trial function $y_0 = Cxe^{-2x} \ \Rightarrow \ y_0' = Ce^{-2x} - 2Cxe^{-2x}$,

$$y_0'' = -4Ce^{-2x} + 4Cxe^{-2x}.$$

So $y_0'' - y_0' - 6y_0 = -5Ce^{-2x} = e^{-2x} \ \Rightarrow \ C = -1/5$

$$\Rightarrow \ \mathrm{PI} = -(1/5)xe^{-2x}.$$

$$\text{Therefore} \ \ y(x) = \mathrm{CF} + \mathrm{PI} = Ae^{3x} + Be^{-2x} - \frac{1}{5}\, xe^{-2x} \ .$$

Sinusoidal h

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

$$h = H\cos x \quad \text{so} \quad Lf \equiv a_2 f'' + a_1 f' + a_0 f = H\cos x$$

$$Lg(x) \equiv a_2 g'' + a_1 g' + a_0 g = H\mathrm{e}^{\mathrm{i}x}$$

$$\Re\mathrm{e}(Lg) = L[\Re\mathrm{e}(g)] = \Re\mathrm{e}(H\mathrm{e}^{\mathrm{i}x}) = H\Re\mathrm{e}(\mathrm{e}^{\mathrm{i}x}) = H\cos x$$

$\Rightarrow$ $\underline{f = \Re\mathrm{e}(g)}$ is the solution to the real equation

Solution: $g = P\mathrm{e}^{\mathrm{i}x}$ $\quad Lg = (-a_2 + \mathrm{i}a_1 + a_0)P\mathrm{e}^{\mathrm{i}x} \quad \Rightarrow \quad P = \dfrac{H}{-a_2 + \mathrm{i}a_1 + a_0}.$

$$\boxed{\begin{aligned} f &= \Re\mathrm{e}\left(\frac{H\mathrm{e}^{\mathrm{i}x}}{(a_0 - a_2) + \mathrm{i}a_1}\right) \\ &= H\frac{(a_0 - a_2)\cos x + a_1 \sin x}{(a_0 - a_2)^2 + a_1^2}. \end{aligned}}$$

Ex 5 $$\boxed{f'' + 3f' + 2f = \cos x.}$$

$$g'' + 3g' + 2g = \mathrm{e}^{\mathrm{i}x}.$$

$$g = P\mathrm{e}^{\mathrm{i}x} \quad \text{where} \quad P = \frac{1}{-1+3\mathrm{i}+2}.$$

PI $$\boxed{f_1 = \Re\mathrm{e}\left(\frac{\mathrm{e}^{\mathrm{i}x}}{1+3\mathrm{i}}\right) = \tfrac{1}{10}(\cos x + 3\sin x).}$$

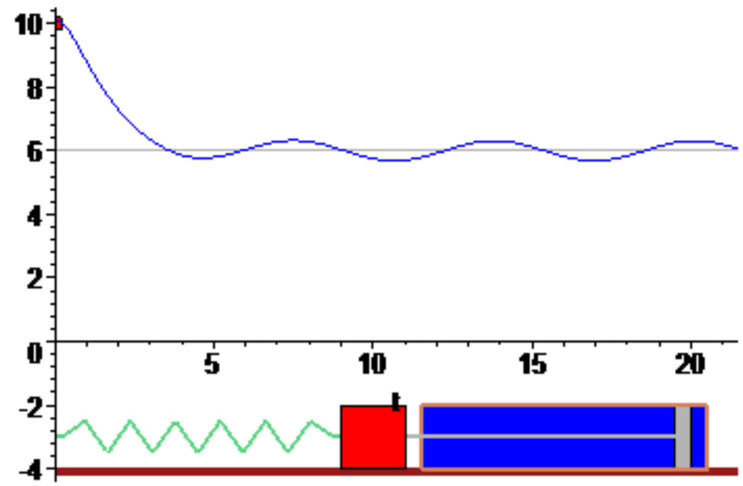
CF $$f_0 = Ae^{-x} + Be^{-2x}$$

General solution $$\boxed{f = Ae^{-x} + Be^{-2x} + \tfrac{1}{10}(\cos x + 3\sin x)}$$

$$f'' + 3f' + 2f = \cos x.$$

$$f = Ae^{-x} + Be^{-2x} + \tfrac{1}{10}(\cos x + 3\sin x)$$

$x(0) = 4, \quad x'(0) = 1 \quad \Rightarrow \quad A = \dfrac{17}{2} \quad B = -\dfrac{23}{5}$



Ex 5

$$\boxed{f'' + 3f' + 2f = \cos x.}$$

What if $\cos(x) \to \sin(x)$?

$$g'' + 3g' + 2g = \mathrm{e}^{ix}.$$

$$f = \Im\mathrm{m}(g)$$

$$g = P\mathrm{e}^{ix} \quad \text{where} \quad P = \frac{1}{-1+3\mathrm{i}+2}.$$

PI

$$\boxed{f_1 = \Im\mathrm{m}\left(\frac{\mathrm{e}^{ix}}{1+3\mathrm{i}}\right) = \tfrac{1}{10}(\sin x - 3\cos x).}$$

CF

$$f_0 = Ae^{-mx} + Be^{-2mx}$$

General solution

$$\boxed{f = Ae^{-mx} + Be^{-2mx} + \tfrac{1}{10}(\sin x - 3\cos x)}$$

Ex 6 $$f'' + 3f' + 2f = 3\cos x + 4\sin x.$$

$$= 5\cos(x+\phi) = 5\Re\mathrm{e}(\mathrm{e}^{\mathrm{i}(x+\phi)})$$

$$\text{where } \phi = \arctan(-4/3)$$

Proof:

$$\cos(x+\phi) = \cos x \cos\phi - \sin x \sin\phi$$

$$A\cos x + B\sin x = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos x + \frac{B}{\sqrt{A^2+B^2}}\sin x\right)$$

$$= \sqrt{A^2+B^2}\cos(x+\phi),$$

$$\cos\phi = A/\sqrt{A^2+B^2}, \quad \sin\phi = -B/\sqrt{A^2+B^2}$$

and

$$\boxed{\tan\phi = -B/A}$$

Ex 6 $$f'' + 3f' + 2f = 3\cos x + 4\sin x.$$

$$= 5\cos(x+\phi) = 5\Re e(\mathrm{e}^{\mathrm{i}(x+\phi)})$$

where $\phi = \arctan(-4/3)$

$$g'' + 3g' + 2g = 5\mathrm{e}^{\mathrm{i}(x+\phi)}$$

Trial solution : $\mathrm{g} = P\mathrm{e}^{\mathrm{i}(x+\phi)}$

$$P = \frac{5}{-1+3\mathrm{i}+2} = \frac{5}{1+3\mathrm{i}},$$

$$f_1 = 5\Re e\left(\frac{\mathrm{e}^{\mathrm{i}(x+\phi)}}{1+3\mathrm{i}}\right) = \tfrac{1}{2}[\cos(x+\phi) + 3\sin(x+\phi)].$$

Ex 7 $\boxed{f'' + f = \cos x} \Rightarrow g'' + g = e^{ix}$

C.F. $(\frac{d}{dx}+i)(\frac{d}{dx}-i)g = e^{ix}$ $\Rightarrow$ $g = Ce^{ix}$, $f = A\cos(x) + B\sin(x)$

P.I. $(\frac{d}{dx}+i)(\frac{d}{dx}-i)g = e^{ix}$.

Try $g = Pxe^{ix}$

Then $e^{ix} = (\frac{d}{dx}+i)(\frac{d}{dx}-i)Pxe^{ix} = (\frac{d}{dx}+i)Pe^{ix} = 2iPe^{ix}$

$$\Rightarrow \quad P = \frac{1}{2i} \quad \Rightarrow \quad \boxed{f = \Re e(\frac{xe^{ix}}{2i}) = \tfrac{1}{2}x\sin x}$$

Ex 8

$$\boxed{f'' + f = \mathrm{e}^{-x}(3\cos x + 4\sin x)} = 5\Re\mathrm{e}(\mathrm{e}^{-x} e^{i(x+\phi)})$$

where $\phi = \arctan(-4/3)$

Trial function

$$g = P\mathrm{e}^{(\mathrm{i}-1)x+\mathrm{i}\phi}$$

$$P = \frac{5}{(\mathrm{i}-1)^2+1} = \frac{5}{1-2\mathrm{i}}.$$

PI $\boxed{f_1 = 5\Re\mathrm{e}\left(\frac{\mathrm{e}^{(\mathrm{i}-1)x+\mathrm{i}\phi}}{1-2\mathrm{i}}\right) = \mathrm{e}^{-x}[\cos(x+\phi) - 2\sin(x+\phi)].}$

Recap

2nd-order linear ODEs with constant coefficients:

$$a_2 f'' + a_1 f' + a_0 f = h(x)$$

- General solution $= \mathrm{PI} + \mathrm{CF}$
  - $\mathrm{CF} = c_1 u_1 + c_2 u_2$,

    u_1 and u_2 linearly independent solutions of the homogeneous equation

▷ Complementary function CF by solving *auxiliary equation*

▷ Particular integral PI by *trial function* with functional form of the inhomogeneous term

♠ Next: physical application to forced, damped oscillator