SECOND-ORDER LINEAR ODEs

$$f'' + p(x)f' + q(x)f = h(x)$$

$\Diamond$ Generalities

$\Diamond$ Structure of general solution

$\Diamond$ Equations with constant coefficients

Second order linear equations

General form : $$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}f}{\mathrm{d}x} + q(x) f = h(x).$$

Integrating factor? Suppose $\exists \quad I(x)$ such that $\frac{\mathrm{d}^2 If}{\mathrm{d}x^2} = Ih$

$$\Rightarrow \qquad 2\frac{\mathrm{d}I}{\mathrm{d}x} = Ip \quad \text{and} \quad \frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = Iq.$$

These equations are incompatible in most cases….

We will study a subset of 2nd order equations which appear in a wide

Structure of the general solution (GS)

$$f'' + p(x)f' + q(x)f = h(x)$$

♠ The general solution f is the sum of a particular solution f_0 (the "particular integral", PI) and the general solution f_1 of the associated homogeneous equation (the "complementary function", CF):

$$f = f_0 + f_1 \ ,$$

$$\text{i.e.,} \quad \text{GS} \ = \ \text{PI} \ + \ \text{CF} \ .$$

♠ The complementary function CF is given by linear combination of two *linearly independent* (↪ see next) solutions u_1 and u_2:

$$\text{CF} = c_1 u_1(x) + c_2 u_2(x)$$

c_1 and c_2 arbitrary constants

♠ Two functions $u_1(x)$ and $u_2(x)$ are *linearly independent* if the relation $\alpha u_1(x) + \beta u_2(x) = 0$ implies $\alpha = \beta = 0$.

Let $\quad \alpha u_1(x) + \beta u_2(x) = 0.$

Differentiating $\Rightarrow \alpha u_1'(x) + \beta u_2'(x) = 0.$

• If $\quad W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_2 u_1' \neq 0$

then $\alpha = \beta = 0$, and u_1 and u_2 are linearly independent.

• If $W(u_1, u_2) = u_1 u_2' - u_2 u_1' = 0$

then $u_2 = \mathrm{constant} \times u_1 \Rightarrow u_1$ and u_2 not linearly independent

$W(u_1, u_2) =$ wronskian determinant of functions u_1 and u_2

♠ n functions $u_1(x), \ldots, u_n(x)$ are linearly independent if

$$\alpha_1 u_1(x) + \ldots + \alpha_n u_n(x) = 0 \implies \alpha_1 = \ldots = \alpha_n = 0.$$

Example: The functions $u_1(x) = \sin x$ and $u_2(x) = \cos x$ are linearly independent.

$$\text{Let} \quad \alpha \sin x + \beta \cos x = 0 \ .$$

$$\text{Differentiating} \ \Rightarrow \ \alpha \cos x - \beta \sin x = 0 \ .$$

$$\text{So} \quad \alpha = \beta \, \frac{\sin x}{\cos x} \ \Rightarrow \ \beta \left(\frac{\sin^2 x}{\cos x} + \cos x \right) = 0 \ \Rightarrow \ \beta \, \frac{1}{\cos x} = 0 \ \Rightarrow \ \beta = 0 \ .$$

Thus $\alpha = \beta = 0$.

- Alternatively:

$$W(u_1, u_2) = u_1 u_2' - u_2 u_1' = - \sin^2 x - \cos^2 x = -1 \ \Rightarrow \ \text{linear independence}$$

Homework

Determine whether the following sets of functions are linearly independent.

$x^2\ ,\ \ x\ ,\ \ 1$ [Answ.: linearly independent]

$1-x\ ,\ \ 1+x\ ,\ \ 1-3x$ [Answ.: linearly dependent]

$e^x\ ,\ \ e^{-x}$ [Answ.: linearly independent]

$e^x\ ,\ \ xe^x\ ,\ \ x^2e^x\ ,\ \ x^3e^x$ [Answ.: linearly independent]

$\sin x\ ,\ \ \cos x\ ,\ \ \sin(x+\alpha)$ [Answ.: linearly dependent]

Homogeneous equation:

$$f'' + p(x) f' + q(x) f = 0$$

♠ Any solution u can be written as a linear combination of linearly independent solutions u_1 and u_2.

Since u, u_1 and u_2 all solve the homogeneous eq., with nonzero coefficients of the second-derivative, first-derivative and no-derivative terms, we must have det $= 0$

$$\begin{vmatrix} u & u_1 & u_2 \\ u' & u_1' & u_2' \\ u'' & u_1'' & u_2'' \end{vmatrix} = 0$$

$W(u, u_1, u_2) = 0$ for solutions u, u_1, u_2

$\Rightarrow \ \alpha u + \beta u_1 + \gamma u_2 = 0$ for α, β and γ not all zero.

Solving for u expresses the solution u as a linear combination of u_1 and u_2.

$$\Rightarrow \ \mathrm{CF} = c_1 u_1(x) + c_2 u_2(x)$$

• solutions span whole set of linear combinations of two independent u_1, u_2

Example: The general solution of the 2nd-order linear ODE

$$y'' + y = 0$$

is $A \sin x + B \cos x$.

(simple harmonic oscillator)

To show this, it is sufficient to show that

i) $\sin x$ and $\cos x$ solve the equation (e.g. by direct computation) and

ii) $\sin x$ and $\cos x$ are linearly independent (see previous Example).

Then general theorem $\mathrm{CF} = c_1 u_1(x) + c_2 u_2(x)$ yields the result.

- Useful reference for this part of the course, with worked problems and examples, is

Schaum's Outline Series

Differential Equations

R. Bronson and G. Costa
McGraw-Hill (Third Edition, 2006)

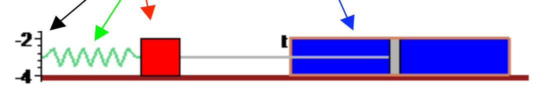
$\Diamond$ See chapters 8 to 14.

2nd-order linear ODEs with constant coefficients:

- general methods of solution available
- arise in many physical applications

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$



Second order linear equation with constant coefficients

Solution

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

The number of independent complementary functions is the number of integration constants – equal to the order of the differential equation

Complementary function

1) Construct f_0 the general solution to the homogeneous equation $Lf_0 = 0$

2) Find a solution, f_1, to the inhomogeneous equation $Lf_1 = h$

Particular integral

General solution : $f_0 + f_1$

For a nth order differential equation need n independent solutions to Lf=0 to specify the complementary function

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

Complementary function

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = 0.$$

Try $y = \mathrm{e}^{mx}$ ⇨ $a_2 m^2 + a_1 m + a_0 = 0.$

$$m_\pm \equiv \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2},$$

"Auxiliary" equation

$a_1^2 - 4a_2 a_0 \rightarrow +, 0, -$

⇨ Complementary function $y = A_+ \mathrm{e}^{m_+ x} + A_- \mathrm{e}^{m_- x}.$

Two constants of integration

Ex 1

$$\boxed{\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=0.}$$

Auxiliary eq. $(m+3)(m+1)=0 \quad \Rightarrow \quad$ CF is $\boxed{y=Ae^{-3x}+Be^{-x}}$

.

$\boxed{\text{If } y(0)=2,\ y'(0)=0} \Rightarrow \quad A+B=2, \quad -3A-B=0$

Initial conditions

A=-1, B=3 $\quad \Rightarrow \quad \boxed{y=-e^{-3x}+3e^{-x}}$

Ex 2

$$\boxed{Ly = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0.}$$

Complex?

Auxiliary eq. $m^2 + 2m + 5 = 0$

i.e. $m = \frac{1}{2}(-2 \pm \sqrt{4-20}) = -1 \pm 2\mathrm{i} \quad \Rightarrow$ CF is $\boxed{y = A\mathrm{e}^{(-1+2\mathrm{i})x} + B\mathrm{e}^{(-1-2\mathrm{i})x}}$

But L is a real operator $\Rightarrow \quad 0 = \Re\mathrm{e}(Ly) = L[\Re\mathrm{e}(y)]$

i.e. $\Re\mathrm{e}(y)$ is a solution (as is $\Im\mathrm{m}(y)$) $\Rightarrow$ $\boxed{\Re\mathrm{e}(y) = \mathrm{e}^{-x}[A'\cos(2x) + B'\sin(2x)].}$

Find the solution for which $y(0) = 5$ and $(\mathrm{d}y/\mathrm{d}x)_0 = 0$

Initial conditions

$5 = A'$

$0 = -A' + 2B' \quad \Rightarrow \quad B' = \frac{5}{2} \quad \Rightarrow \quad \boxed{y = \mathrm{e}^{-x}[5\cos(2x) + \frac{5}{2}\sin(2x)].}$

Factorisation of operators

We wish to solve $$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = 0.$$

We did this by trying $y = \mathrm{e}^{mx}$

$$\Rightarrow \quad a_2 m^2 + a_1 m + a_0 = a_2 (m - m_+)(m - m_-) = 0. \qquad m_\pm = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

This is equivalent to factorising the equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - m_-\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - m_+\right) f = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - (m_- + m_+)\frac{\mathrm{d}f}{\mathrm{d}x} + m_- m_+ f$$
$$= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{a_1}{a_2}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{a_0}{a_2} \equiv \frac{Lf}{a_2}$$

Now we can see why the CF is made up of exponentials …. because :

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - m_-\right)\mathrm{e}^{m_- x} = 0 \quad ; \quad \left(\frac{\mathrm{d}}{\mathrm{d}x} - m_+\right)\mathrm{e}^{m_+ x} = 0.$$

Factorisation of operators and repeated roots

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = 0.$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - m_-\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - m_+\right) f = 0 \qquad m_\pm = \frac{-a_1 \pm \sqrt{a_1^{\,2} - 4a_2 a_0}}{2a_2}$$

What happens if $a_1^2 - 4a_2 a_0 = 0$ and $m_+ = m_- = m$?

$$Lf = \left(\frac{\mathrm{d}}{\mathrm{d}x} - m\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - m\right) f.$$

$\boxed{e^{mx}}$ gives one solution $: L(\mathrm{e}^{mx}) = \left(\frac{\mathrm{d}}{\mathrm{d}x} - m\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - m\right)\mathrm{e}^{mx} = 0$

$\boxed{xe^{mx}}$ gives the second independent solution :

$$L(x\mathrm{e}^{mx}) = \left(\frac{\mathrm{d}}{\mathrm{d}x} - m\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - m\right)x\mathrm{e}^{mx} = \left(\frac{\mathrm{d}}{\mathrm{d}x} - m\right)\mathrm{e}^{mx} = 0,$$

$$\boxed{y = A\mathrm{e}^{mx} + Bx\mathrm{e}^{mx}}$$

Ex 4

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

Auxiliary equation $(m+1)^2 = 0$

$$\Rightarrow \boxed{y = A\mathrm{e}^{-x} + Bx\mathrm{e}^{-x}}$$

Ex 5

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} - 2\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

Auxiliary equation $(m-1)^2(m-\mathrm{i})(m+\mathrm{i}) = 0$

$$\Rightarrow \boxed{y = \mathrm{e}^x(A+Bx) + C\cos x + D\sin x.}$$

Summary

2nd order linear ODEs: $f'' + p(x)f' + q(x)f = h(x)$

- General solution $=$ PI $+$ CF
- $\mathrm{CF} = c_1 u_1 + c_2 u_2$

u_1 and u_2 linearly independent solutions of the homogeneous equation

- Equations with constant coefficients:

$\triangleright$ Solve auxiliary equation to find complementary function CF

$\diamondsuit$ distinct real roots

$\diamondsuit$ repeated roots

$\diamondsuit$ complex roots

$\triangleright$ Next: methods to find the particular integral PI